Chapter 20

Analysis of MIMO Control Loops

Motivational Examples

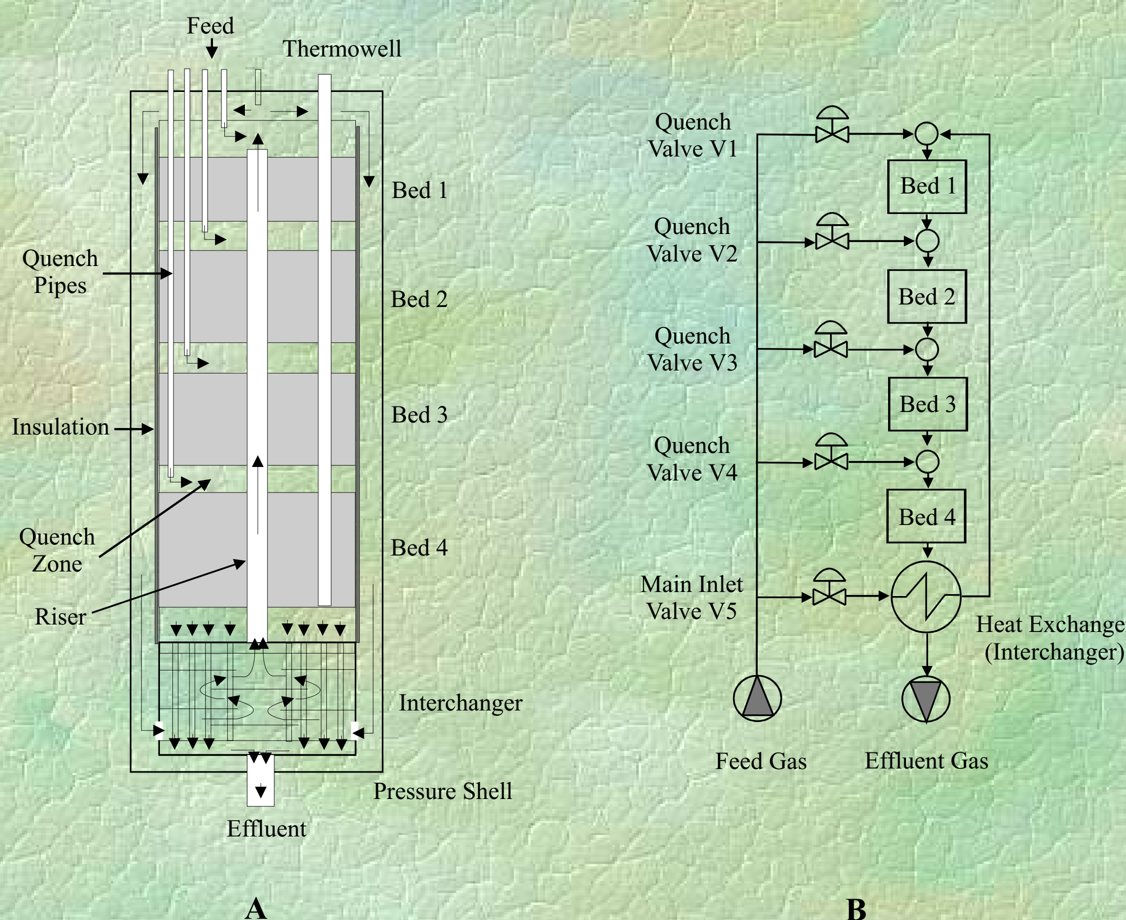
All real-world systems comprise multiple interacting variables. For example, one tries to increase the flow of water in a shower by turning on the hot tap, but then the temperature goes up; one wants to spend more time on holiday, but then one needs to spend more time at work to earn more money. A more physical example is provided on the next slide.

Example 20.1 (Ammonia Plant)

A typical industrial plant aimed at producing ammonia from natural gas is the Kellogg Process.

In an integrated chemical plant of this type, there will be hundreds (*possibly thousands*) of variables that interact to some degree. Even if one focuses on one particular process unit - e.g., the ammonia synthesis converters - one still ends up with 5 to 10 highly coupled variables. A typical ammonia-synthesis converter is shown below.

Figure 20.1: *Ammonia-synthesis converter*



The process is exothermic; thus, the temperature rises across each catalyst bed. It is then cooled by mixing from the quench flows. Many measurements will typically be made - e.g., the temperature on either side of each bed.

The nature of the interactions can be visualized as follows. Say one incrementally opens quench valve 1; then all other flows will be affected, the temperature in zone 1 will drop, this will pass down the converter from bed to bed; as the reaction progressively slows, the heat exchanger will move to a different operating point and finally, the temperature of the feed into the top of the converter will be affected. Thus, in the end, all variables will respond to the change in a single manipulated variable.

Obviously, these kinds of interaction are complex to understand and, as a result, they make control-system design *interesting*. Of course, one could attempt to solve the problem by using several SISO control loops, but this might not prove satisfactory. For example, in the ammonia-synthesis plant one could try controlling T_1, T_3, T_5, and T_7 by manipulating the four quench valves with individual PID controllers. However, this turns out to be a somewhat nontrivial task, on account of the associated interactions.

Models for Multivariable Systems

Most of the ideas presented in early parts of the book apply (*albeit with some slight enhancements*) to multivariable systems. The main difficulty in the MIMO case is that we have to work with matrix, rather than scalar transfer functions. This means that care needs to be taken with such issues as the order in which transfer functions appear. (*In general matrices do not commute*).

State Space Models, Revisited

Linear MIMO systems can be described by using the state space ideas presented in Chapter 17. The only change is the extension of the dimensions of inputs and outputs to vectors.

Transfer-Function Models, Revisited

It is straightforward to convert a state space model to a transfer-function model.

The matrix transfer function $\mathbf{G}(s)$ corresponding to a state space model $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is

$$\mathbf{G}(s) \stackrel{\triangle}{=} \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

We will use $G_{ik}(s)$ to denote the transfer function from the k^{th} component of $U(s)$ to the i^{th} component of $Y(s)$. Then $\mathbf{G}(s)$ can be expressed as

$$\mathbf{G}(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \dots & G_{1k}(s) & \dots & G_{1m}(s) \\ G_{21}(s) & G_{22}(s) & \dots & G_{2k}(s) & \dots & G_{2m}(s) \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ G_{i1}(s) & G_{i2}(s) & \dots & G_{ik}(s) & \dots & G_{im}(s) \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ G_{m1}(s) & G_{m2}(s) & \dots & G_{mk}(s) & \dots & G_{mm}(s) \end{bmatrix}$$

Definition 20.2: The *impulse response matrix* of the system, $\mathbf{g}(t)$, is the inverse Laplace transform of the transfer-function matrix $\mathbf{G}(s)$. For future reference, we express $\mathbf{g}(t)$ as

$$\mathbf{g}(t) = \begin{bmatrix} g_{11}(t) & g_{12}(t) & \dots & g_{1k}(t) & \dots & g_{1m}(t) \\ g_{21}(t) & g_{22}(t) & \dots & g_{2k}(t) & \dots & g_{2m}(t) \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ g_{i1}(t) & g_{i2}(t) & \dots & g_{ik}(t) & \dots & g_{im}(t) \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ g_{m1}(t) & g_{m2}(t) & \dots & g_{mk}(t) & \dots & g_{mm}(t) \end{bmatrix} = \mathcal{L}^{-1}\left[\mathbf{G}(s)\right]$$

Matrix Fraction Descriptions

Clearly, all matrix transfer descriptions comprise elements having numerator and denominator polynomials. These matrices of rational functions of polynomials can be factorized in various ways.

Left Matrix Fraction Description (LMFD)

We can write $\mathbf{G}(s) = \left[\bar{G}_D(s)\right]^{-1} \left[\bar{G}_N(s)\right]$ where

$$\bar{\mathbf{G}}_D(s) = \begin{bmatrix} \frac{d_1^r(s)}{e_1(s)} & & \\ & \ddots & \\ & & \frac{d_m^r(s)}{e_m(s)} \end{bmatrix}$$

$$\bar{\mathbf{G}}_N(s) = \begin{bmatrix} \frac{n_{11}(s)}{e_1(s)} & \dots & \frac{n_{1m}(s)}{e_1(s)} \\ \vdots & & \\ \frac{n_{m1}(s)}{e_m(s)} & \dots & \frac{n_{mm}(s)}{e_m(s)} \end{bmatrix}$$

This is a special form of Left Matrix Fraction Description (LMFD) for $\mathbf{G}(s)$.

Right Matrix Fraction Description (RMFD)

Let $d_i^c(s)$ denote the least common multiple of the denominator polynomials in the i^{th} column of $\mathbf{G}(s)$. Also, let $e_i'(s)$ denote a Hurwitz polynomial of the same degree as $d_i^c(s)$ Then we can write

$$\mathbf{G}(s) = [G_N(s)]\,[G_D(s)]^{-1}$$

where

$$\mathbf{G}_D(s) = \begin{bmatrix} \frac{d_1^c(s)}{e_1'(s)} & & \\ & \ddots & \\ & & \frac{d_m^c(s)}{e_m'(s)} \end{bmatrix}$$

$$\mathbf{G}_N(s) = \begin{bmatrix} \frac{n_{11}'(s)}{e_1'(s)} & \dots & \frac{n_{1m}'(s)}{e_m'(s)} \\ \vdots & & \\ \frac{n_{m1}'(s)}{e_1'(s)} & \dots & \frac{n_{mm}'(s)}{e_m'(s)} \end{bmatrix}$$

This is a special form of Right Matrix Fraction Description.

Connection Between State Space Models and MFD's

A RMFD and LMFD can be obtained from a state space description of a given system by designing stabilizing state-variable feedback and an observer, respectively.

Consider the state space model

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}x(t)$$

We assume that the state space model is stabilizable.

Let $u(t) = -\mathbf{K}x(t) + w(t)$ be stabilizing feedback. The system can then be written as follows, by adding and subtracting $\mathbf{BK}x(t)$:

$$\dot{x}(t) = (\mathbf{A} - \mathbf{BK})x(t) + \mathbf{B}w(t)$$
$$y(t) = \mathbf{C}x(t)$$
$$w(t) = u(t) + \mathbf{K}x(t)$$

We can express these equations, in the Laplace-transform domain with zero initial conditions, as

$$U(s) = (\mathbf{I} - \mathbf{K}[s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}]^{-1}\mathbf{B})W(s)$$
$$Y(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}]^{-1}\mathbf{B}W(s)$$

$$U(s) = \mathbf{G_D}(s)W(s); \qquad Y(s) = \mathbf{G_N}(s)W(s); \qquad Y(s) = \mathbf{G_N}(s)[\mathbf{G_D}(s)]^{-1}U(s$$

where $\mathbf{G_N}(s)$ and $\mathbf{G_D}(s)$ are the following two stable transfer-function matrices:

$$\mathbf{G_N}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}]^{-1}\mathbf{B}$$
$$\mathbf{G_D}(s) = \mathbf{I} - \mathbf{K}[s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}]^{-1}\mathbf{B}$$

We see that $(\mathbf{G_N}(s), \mathbf{G_D}(s))$ is a RMFD.

Similarly, we can use an observer to develop a LMFD. We assume that the state space model is detectable. Consider the following observer

$$\dot{\hat{x}}(t) = \mathbf{A}\hat{x}(t) + \mathbf{B}u(t) + \mathbf{J}(y(t) - \mathbf{C}\hat{x}(t))$$
$$y(t) = \mathbf{C}\hat{x}(t) + \nu(t)$$

We can express these equations in the Laplace domain as

$$\Phi(s) \stackrel{\triangle}{=} \mathcal{L}\left[\nu(t)\right] = (\mathbf{I} - \mathbf{C}[s\mathbf{I} - \mathbf{A} + \mathbf{J}\mathbf{C}]^{-1}\mathbf{J})Y(s) - \mathbf{C}[s\mathbf{I} - \mathbf{A} + \mathbf{J}\mathbf{C}]^{-1}\mathbf{B}U(s)$$

We know that, for a stable observer, $v(t) \rightarrow 0$ exponentially fast, hence, in steady state, we can write

$$\overline{\mathbf{G}}_{\mathbf{D}}(s)Y(s) = \overline{\mathbf{G}}_{\mathbf{N}}(s)U(s)$$

where

$$\overline{\mathbf{G}}_{\mathbf{N}}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A} + \mathbf{JC})^{-1}\mathbf{B}$$

$$\overline{\mathbf{G}}_{\mathbf{D}}(s) = \mathbf{I} - \mathbf{C}(s\mathbf{I} - \mathbf{A} + \mathbf{JC})^{-1}\mathbf{J}$$

Hence $(\overline{\mathbf{G}}_{\mathbf{N}}(s), \overline{\mathbf{G}}_{\mathbf{D}}(s))$ is a LMFD for the system.

The RMFD and LMFD developed above have the following interesting property:

Lemma 20.1: There always exist a RMFD and a LMFD for a system having the following coprime factorization property:

$$\begin{bmatrix} \overline{\mathbf{C}}_{\mathbf{D}}(s) & \overline{\mathbf{C}}_{\mathbf{N}}(s) \\ -\overline{\mathbf{G}}_{\mathbf{N}}(s) & \overline{\mathbf{G}}_{\mathbf{D}}(s) \end{bmatrix} \begin{bmatrix} \mathbf{G}_{\mathbf{D}}(s) & -\mathbf{C}_{\mathbf{N}}(s) \\ \mathbf{G}_{\mathbf{N}}(s) & \mathbf{C}_{\mathbf{D}}(s) \end{bmatrix} =$$

$$\begin{bmatrix} \mathbf{G}_{\mathbf{D}}(s) & -\mathbf{C}_{\mathbf{N}}(s) \\ \mathbf{G}_{\mathbf{N}}(s) & \mathbf{C}_{\mathbf{D}}(s) \end{bmatrix} \begin{bmatrix} \overline{\mathbf{C}}_{\mathbf{D}}(s) & \overline{\mathbf{C}}_{\mathbf{N}}(s) \\ -\overline{\mathbf{G}}_{\mathbf{N}}(s) & \overline{\mathbf{G}}_{\mathbf{D}}(s) \end{bmatrix} = \mathbf{I}$$

Where $\mathbf{G_N}(s), \mathbf{G_D}(s)$ etc. are defined by

$$\mathbf{G_N}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A} + \mathbf{BK}]^{-1}\mathbf{B}$$
$$\mathbf{G_D}(s) = \mathbf{I} - \mathbf{K}[s\mathbf{I} - \mathbf{A} + \mathbf{BK}]^{-1}\mathbf{B}$$
$$\overline{\mathbf{G}}_\mathbf{N}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A} + \mathbf{JC})^{-1}\mathbf{B}$$
$$\overline{\mathbf{G}}_\mathbf{D}(s) = \mathbf{I} - \mathbf{C}(s\mathbf{I} - \mathbf{A} + \mathbf{JC})^{-1}\mathbf{J}$$
$$\mathbf{C_N}(s) = \mathbf{K}[s\mathbf{I} - \mathbf{A} + \mathbf{BK}]^{-1}\mathbf{J}$$
$$\mathbf{C_D}(s) = \mathbf{I} + \mathbf{C}[s\mathbf{I} - \mathbf{A} + \mathbf{BK}]^{-1}\mathbf{J}$$
$$\overline{\mathbf{C}}_\mathbf{N}(s) = \mathbf{K}[s\mathbf{I} - \mathbf{A} + \mathbf{JC}]^{-1}\mathbf{J}$$
$$\overline{\mathbf{C}}_\mathbf{D}(s) = \mathbf{I} + \mathbf{K}[s\mathbf{I} - \mathbf{A} + \mathbf{JC}]^{-1}\mathbf{B}$$

Poles and Zeros of MIMO Systems

The reader will recall that, in the SISO case, the performance of control systems was markedly dependent on the location of open-loop zeros. Thus, it would seem to be important to extend the notion of zeros to the MIMO case.

We define zeros of a MIMO transfer function as those values of *s* that make the matrix **G**(*s*) lose rank. This means that there exists at least one nonzero constant vector *v* (*zero right direction*) such that

$$\mathbf{G}(c)v = 0$$

and at least one nonzero constant vector *w* (*zero left direction*) such that

$$w^T\mathbf{G}(c) = 0$$

where *s* = *c* is one of the zeros of **G**(*s*).

Note that the number of linearly independent vectors that satisfy $\mathbf{G}(c)v = 0$ depends on the rank loss of **G**(*s*) when evaluated at *s* = *c*. This number if known as the *geometric multiplicity* of the zero, and it is equal to the dimension of the null space generated by the columns of **G**(*s*).

System zeros as defined above are not always obvious by looking at the transfer function. This is illustrated in the following example.

Example 20.3

Consider the matrix transfer function

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{4}{(s+1)(s+2)} & \dfrac{-1}{(s+1)} \\ \dfrac{2}{(s+1)} & \dfrac{-1}{2(s+1)(s+2)} \end{bmatrix}$$

It is difficult to tell by inspection where its zeros are. However, it turns out there is one zero at $s = -3$, as can be readily seen by noting that

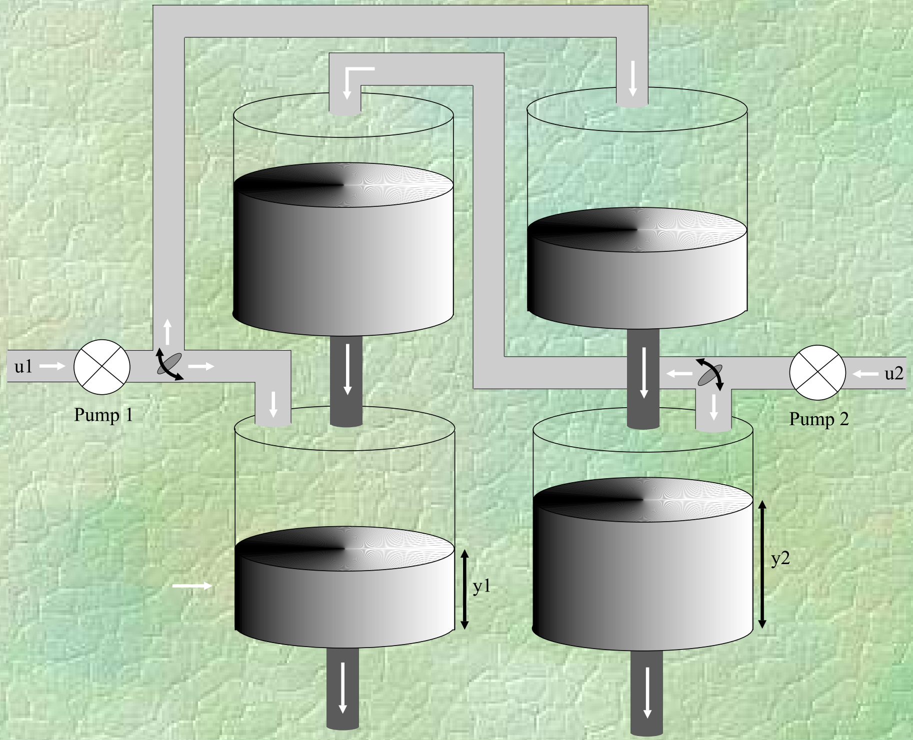
$$\mathbf{G}(-3) = \begin{bmatrix} 2 & \dfrac{1}{2} \\ -1 & -\dfrac{1}{4} \end{bmatrix}$$

which clearly has rank 1.

Example 20.4: Quadruple-tank Apparatus

A very interesting piece of laboratory equipment based on four coupled tanks is shown in the next photo.

Figure 20.2: *Schematic of a quadruple-tank apparatus*



Physical modeling leads to the following (*linearized*) transfer function linking (u_1, u_2) with (y_1, y_2).

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{3.7\gamma_1}{62s+1} & \dfrac{3.7(1-\gamma_2)}{(23s+1)(62s+1)} \\ \dfrac{4.7(1-\gamma_1)}{(30s+1)(90s+1)} & \dfrac{4.7\gamma_2}{90s+1} \end{bmatrix}$$

Where γ_1 and (1 - γ_1) represent the proportion of the flow from pump 1 that goes into tanks 1 and 4, respectively (*similarly for* γ_2 *and* (1 - γ_2).

The system has two multivariable zeros that satisfy $\det(\mathbf{G}(s)) = 0$:

$$(23s+1)(30s+1) - \eta = 0 \qquad \text{where} \quad \eta = \frac{(1-\gamma_1)(1-\gamma_2)}{\gamma_1\gamma_2}$$

A simple root-locus argument shows that the system is nonminimum phase for $\eta > 1$, i.e. for $0 < \gamma_1 + \gamma_2 < 1$, and minimum phase for $\eta < 1$, i.e. for $1 < \gamma_1 + \gamma_2 < 2$.

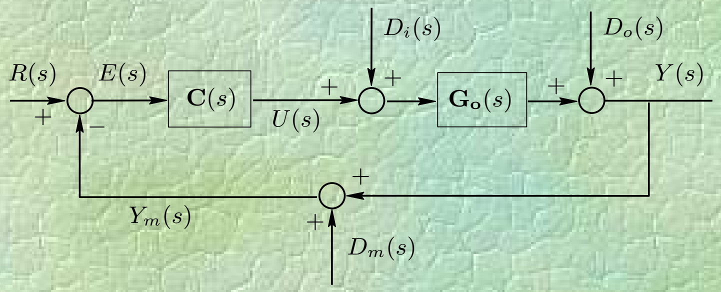
Also, the zero direction associated with a zero $c > 0$ satisfies $w^T\mathbf{G}(c) = 0$. It then follows that, if γ_1 is small, the zero is associated mostly with the first output, whilst if γ_1 is close to 1, then the zero is associated mostly with the second output.

The Basic MIMO Control Loop

The systems we consider will be square (*the input vector has the same number of components as the output vector*). Also, all transfer-function matrices under study will be assumed to be nonsingular almost everywhere, which means that these matrices will be singular only at a finite set of *zeros*.

We will consider the same basic feedback structure as in the SISO case, i.e., the structure shown on the next slide.

Figure 20.3: *MIMO feedback loop*



The nominal MIMO control loop can be described, as in the SISO case, by certain key transfer functions. In particular, we define

$S_0(s)$: the (*matrix*) transfer function connecting $D_0(s)$ to $Y(s)$

$T_0(s)$: the (*matrix*) transfer function connecting $R(s)$ to $Y(s)$

$S_{u0}(s)$: the (*matrix*) transfer function connecting $R(s)$ to $U(s)$

$S_{i0}(s)$: the (*matrix*) transfer function connecting $D_i(s)$ to $Y(s)$

$$Y(s) = \mathbf{T_o}(s)R(s) - \mathbf{T_o}(s)D_m(s) + \mathbf{S_o}(s)D_o(s) + \mathbf{S_{io}}(s)D_i(s)$$
$$U(s) = \mathbf{S_{uo}}(s)R(s) - \mathbf{S_{uo}}(s)D_m(s) - \mathbf{S_{uo}}(s)D_o(s) - \mathbf{S_{uo}}(s)\mathbf{G_o}(s)D_i(s)$$
$$E(s) = \mathbf{S_o}(s)R(s) - \mathbf{S_o}(s)D_m(s) - \mathbf{S_o}(s)D_o(s) - \mathbf{S_{io}}(s)D_i(s)$$

MIMO Sensitivity Functions

The Sensitivity Functions (*used in the expressions on the previous slide*) are given by

$$
\begin{aligned}
\mathbf{S_o}(s) &= [\mathbf{I} + \mathbf{G_o}(s)\mathbf{C}(s)]^{-1} \\
\mathbf{T_o}(s) &= \mathbf{G_o}(s)\mathbf{C}(s)[\mathbf{I} + \mathbf{G_o}(s)\mathbf{C}(s)]^{-1} = [\mathbf{I} + \mathbf{G_o}(s)\mathbf{C}(s)]^{-1}\mathbf{G_o}(s)\mathbf{C}(s) \\
&= \mathbf{I} - \mathbf{S_o}(s) \\
\mathbf{S_{uo}}(s) &= \mathbf{C}(s)[\mathbf{I} + \mathbf{G_o}(s)\mathbf{C}(s)]^{-1} = \mathbf{C}(s)\mathbf{S_o}(s) = [\mathbf{G_o}(s)]^{-1}\mathbf{T_o}(s) \\
\mathbf{S_{io}}(s) &= [\mathbf{I} + \mathbf{G_o}(s)\mathbf{C}(s)]^{-1}\mathbf{G_o}(s) = \mathbf{G_o}(s)[\mathbf{I} + \mathbf{C}(s)\mathbf{G_o}(s)]^{-1} = \mathbf{S_o}(s)\mathbf{G_o}(s)
\end{aligned}
$$

Note that, because matrix products, in general, do not commute, special care must be exercised when manipulating the above equations.

Note also that $\mathbf{S_o}(s) + \mathbf{T_o}(s) = \mathbf{I}$ and $\mathbf{S}(s) + \mathbf{T}(s) = \mathbf{I}$. There are also multivariable versions of

$$S_o(s) + T_o(s) = 1$$

$$S_{io}(s) = S_o(s)G_o(s) = \frac{T_o(s)}{C(s)}$$

$$S_{uo}(s) = S_o(s)C(s) = \frac{T_o(s)}{G_o(s)}$$

Closed-Loop Stability

We next extend the notions of stability, described in Chapter 15 for the SISO case, to the MIMO case.

Consider the nominal control loop in Figure 20.3. Then the nominal loop is internally stable if and only if the four sensitivity functions are stable.

Stability in MFD Form

Stability can also be expressed by using matrix fraction descriptions (MFDs).

Consider RMFD and LMFD descriptions for the plant and the controller:

$$\mathbf{G_o}(s) = \mathbf{G_{oN}}(s)[\mathbf{G_{oD}}(s)]^{-1} = \left[\overline{\mathbf{G}}_{\mathbf{oD}}(s)\right]^{-1}\overline{\mathbf{G}}_{\mathbf{oN}}(s)$$

$$\mathbf{C}(s) = \mathbf{C_N}(s)[\mathbf{C_D}(s)]^{-1} = \left[\overline{\mathbf{C}}_{\mathbf{D}}(s)\right]^{-1}\overline{\mathbf{C}}_{\mathbf{N}}(s)$$

The transfer functions appearing in the sensitivity functions can be rewritten

$$\mathbf{S_o}(s) = \mathbf{C_D}(s)\left[\overline{\mathbf{G}}_{\mathbf{oD}}(s)\mathbf{C_D}(s) + \overline{\mathbf{G}}_{\mathbf{oN}}(s)\mathbf{C_N}(s)\right]^{-1}\overline{\mathbf{G}}_{\mathbf{oD}}(s)$$

$$\mathbf{T_o}(s) = \mathbf{G_{oN}}(s)\left[\overline{\mathbf{C}}_{\mathbf{D}}(s)\mathbf{G_{oD}}(s) + \overline{\mathbf{C}}_{\mathbf{N}}(s)\mathbf{G_{oN}}(s)\right]^{-1}\overline{\mathbf{C}}_{\mathbf{N}}(s)$$

$$\mathbf{S_{uo}}(s) = \mathbf{C_N}(s)\left[\overline{\mathbf{G}}_{\mathbf{oD}}(s)\mathbf{C_D}(s) + \overline{\mathbf{G}}_{\mathbf{oN}}(s)\mathbf{C_N}(s)\right]^{-1}\overline{\mathbf{G}}_{\mathbf{oD}}(s)$$

$$\mathbf{S_{io}}(s) = \mathbf{C_D}(s)\left[\overline{\mathbf{G}}_{\mathbf{oD}}(s)\mathbf{C_D}(s) + \overline{\mathbf{G}}_{\mathbf{oN}}(s)\mathbf{C_N}(s)\right]^{-1}\overline{\mathbf{G}}_{\mathbf{oN}}(s)$$

$$\mathbf{S_{uo}}(s)\mathbf{G_o}(s) = \mathbf{C_N}(s)\left[\overline{\mathbf{G}}_{\mathbf{oD}}(s)\mathbf{C_D}(s) + \overline{\mathbf{G}}_{\mathbf{oN}}(s)\mathbf{C_N}(s)\right]^{-1}\overline{\mathbf{G}}_{\mathbf{oN}}(s)$$

The above expressions immediately imply the result on the next slide.

Stability of Feedback Loops Described via MFD's

Consider a one-d.o.f. MIMO feedback control loop, as shown in Figure 20.3. Let the nominal plant model and the controller be expressed in MFD. Then the nominal loop is internally stable if and only if the closed-loop characteristic matrix $\mathbf{A_{cl}}(s)$

$$\mathbf{A_{cl}}(s) \stackrel{\triangle}{=} \overline{\mathbf{G}}_{\mathbf{oD}}(s)\mathbf{C_D}(s) + \overline{\mathbf{G}}_{\mathbf{oN}}(s)\mathbf{C_N}(s)$$

has all its zeros strictly in the LHP, where the zeros are defined to be the zeros of $\det\{\mathbf{A_{cl}}(s)\}$.

Example 20.5

A diagonal controller $\mathbf{C}(s)$ is proposed to control a MIMO plant with nominal model $\mathbf{G_o}(s)$. If $\mathbf{C}(s)$ and $\mathbf{G_o}(s)$ are given by

$$\mathbf{G_o}(s) = \begin{bmatrix} \dfrac{2}{s+1} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{1}{(s+1)(s+2)} & \dfrac{2}{s+2} \end{bmatrix}; \qquad \mathbf{C}(s) = \begin{bmatrix} \dfrac{2}{s} & 0 \\ 0 & \dfrac{1}{s} \end{bmatrix}$$

is the loop stable?

We need LMFD and RMFD for the plant model and the controller, respectively.

A simple choice is

$$\overline{\mathbf{G}}_{\mathbf{oN}}(s) = \begin{bmatrix} 2(s+2) & 1 \\ 1 & 2(s+1) \end{bmatrix}; \qquad \overline{\mathbf{G}}_{\mathbf{oD}}(s) = (s+1)(s+2)\mathbf{I}$$

and

$$\mathbf{C}_{\mathbf{N}}(s) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}; \qquad \mathbf{C}_{\mathbf{D}}(s) = s\mathbf{I}$$

Then

$$\begin{aligned} \mathbf{A}_{\mathbf{cl}}(s) &= \overline{\mathbf{G}}_{\mathbf{oD}}(s)\mathbf{C}_{\mathbf{D}}(s) + \overline{\mathbf{G}}_{\mathbf{oN}}(s)\mathbf{C}_{\mathbf{N}}(s) \\ &= \begin{bmatrix} 2s^3 + 10s^2 + 18s + 8 & s^2 + 3s + 2 \\ s^2 + 3s + 2 & 2s^3 + 8s^2 + 11s + 4 \end{bmatrix} \end{aligned}$$

$$\det\left(\mathbf{A_{cl}}(s)\right) = 4s^6 + 36s^5 + 137s^4 + 272s^3 + 289s^2 + 148s + 28$$

All roots of det($\mathbf{A_{cl}}(s)$) have negative real parts. Thus, the loop is stable.

Stability via Frequency Responses

The reader may well wonder whether tools from SISO analysis can be applied to test stability for MIMO systems. The answer is, in general, *yes*, but significant complications arise due to the multivariable nature of the problem. We will illustrate by showing how Nyquist theory might be extended to the MIMO case.

If we assume that only stable pole-zero cancellations occur in a MIMO feedback loop, then the internal stability of the nominal loop is ensured by demanding that $S_0(s)$ be stable.

Consider now the function $F_0(s)$, defined as

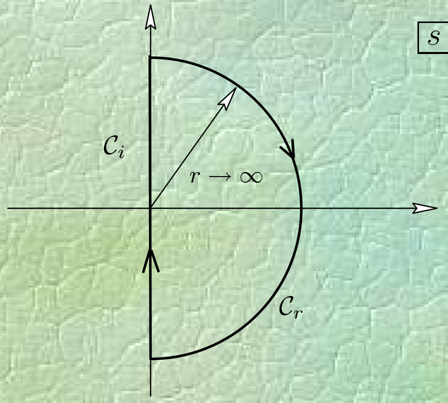
$$F_o(s) = \det(\mathbf{I} + \mathbf{G_o}(s)\mathbf{C}(s)) = \prod_{i=1}^{m}(1 + \lambda_i(s))$$

where $\lambda_i(s)$, $i = 1, 2, \ldots, m$, are the eigenvalues of $\mathbf{G_o}(s)\mathbf{C}(s)$. The polar plots of $\lambda_i(j\ )$, $i = 1, 2, \ldots, m$ on the complex plane are known as characteristic loci.

Stability Result

If the Nyquist contour $C_s = C_i \cup C_r$, shown in Figure 5.5 on the next slide, is chosen, then we have the following theorem, which has been adapted from the Nyquist Theorem.

Figure 5.5



Theorem 10.1: If a proper open-loop transfer function $\mathbf{G_o}(s)\mathbf{C}(s)$ has P poles in the open RHP, then the closed loop has Z poles in the open RHP if and only if the polar plot that describes the combination of all characteristic loci (*along the modified Nyquist path*) encircles the point (-1, 0) clockwise $N = Z\text{-}P$ times.

Proof: See the book.

Steady-State Response for Step Inputs

Steady-state responses also share much in common with the SISO case. Here, however, we have vector inputs and outputs. Thus, we will consider step inputs coming from particular directions - i.e., applied to various combinations of inputs in the input vectors. This is achieved by defining

$$R(s) = K_r \frac{1}{s}; \qquad D_i(s) = K_{di} \frac{1}{s}; \qquad D_o(s) = K_{do} \frac{1}{s}$$

where $K_r \in \mathbb{R}^m$, $K_{di} \in \mathbb{R}^m$, and $K_{d0} \in \mathbb{R}^m$ are constant vectors.

By using the final-value theorem, we have

$$\lim_{t\to\infty} y(t) = \mathbf{T_o}(0)K_r + \mathbf{S_o}(0)K_{do} + \mathbf{S_{io}}(0)K_{di}$$
$$\lim_{t\to\infty} u(t) = \mathbf{S_{uo}}(0)K_r - \mathbf{S_{uo}}(0)K_{do} - \mathbf{S_{uo}}(0)\mathbf{G_o}(0)K_{di}$$
$$\lim_{t\to\infty} e(t) = \mathbf{S_o}(0)K_r - \mathbf{S_o}(0)K_{do} - \mathbf{S_{io}}(0)K_{di}$$

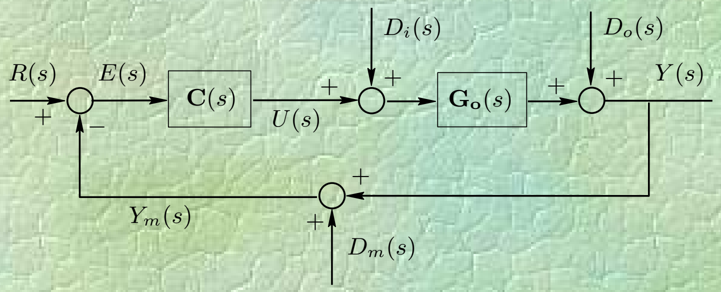
It is also possible to examine the circumstances that lead to zero steady-state errors. By way of illustration, we have the following result for the case of step reference signals.

Consider a stable MIMO feedback loop, as on the next slide. Assume that the reference $R(s)$ is a vector of the form shown in

$$R(s) = K_r \frac{1}{s}; \qquad D_i(s) = K_{di} \frac{1}{s}; \qquad D_o(s) = K_{do} \frac{1}{s}$$

The steady-state error in the i^{th} channel, $e_i(\infty)$, is zero if the i^{th} row of $\mathbf{S_o}(0)$ is zero. Under these conditions, the i^{th} row of $\mathbf{T_o}(0)$ is the elementary vector $e_i = [0 \dots 0\ 1\ 0 \dots 0]^T$.

Figure 20.3



Frequency-Domain Analysis

We found, in the SISO case, that the frequency domain gave valuable insights into the response of a closed loop to various inputs. This is also true in the MIMO case. However, to apply these tools, we need to extend the notion of *frequency-domain gain* to the multivariable case.

Principal Gains and Principal Directions

Consider a MIMO system with m inputs and m outputs, having an $m \times m$ matrix transfer function $\mathbf{G}(s)$:

$$Y(s) = \mathbf{G}(s)U(s)$$

We obtain the corresponding frequency response by setting $s = j\omega$. This leads to the question: *How can one define the* gain *of a MIMO system in the frequency domain ?*

We use vector norms instead of absolute values. Any suitable norm could be used. We will use $||r||$ to denote the norm of the vector υ. For example, we could use the Euclidean norm, defined as follows:

$$||v|| = \sqrt{|v_1|^2 + |v_2|^2 + \ldots |v_n|^2} = \sqrt{v^H v}$$

where υ^H denotes conjugate transpose.

MIMO Gain

A possible way to define the MIMO system gain at frequency ω is then to choose a norm for the matrix **G** that accounts for the maximizing direction associated with the input for U. Thus, we define

$$||\mathbf{G}|| = \sup_{||U|| \neq 0} \frac{||\mathbf{G}U||}{||U||}$$

We call ||**G**|| the induced norm on **G** corresponding to the vector norm ||U||. For example, when the vector norm is chosen to be the Euclidean norm,

$$||x|| = \sqrt{x^H x}$$

then we have the *induced spectral norm* for **G** defined by

$$||\mathbf{G}|| = \sup_{||U|| \neq 0} \frac{||\mathbf{G}U||}{||U||} = \sup_{||U|| \neq 0} \sqrt{\frac{U^H \mathbf{G^H} \mathbf{G} U}{U^H U}}$$

Actually, the above notion of induced norm is closely connected to the notion of singular values. To show this connection, we recall the definition of singular values of an $m \times l$ complex matrix $\mathbf{\Gamma}$. The set of singular values of $\mathbf{\Gamma}$ is a set of cardinality $k = \min(l, m)$ defined by

$$(\sigma_1, \sigma_2, \ldots, \sigma_k) = \begin{cases} \sqrt{\text{eigenvalues of} \quad \mathbf{\Gamma^H \Gamma}} & \text{if } m < l \\ \sqrt{\text{eigenvalues of} \quad \mathbf{\Gamma \Gamma^H}} & \text{if } m \geq l \end{cases}$$

We note that the singular values are real positive values, because $\Gamma^H\Gamma$ and Γ^H are Hermitian matrices. We recall that $\Omega(j\omega)$ is a Hermitian matrix if $\Omega^H(j\omega) \triangleq \Omega^T(-j\omega) = \Omega(j\omega)$. It is customary to order the singular values, as follows:

$$\sigma_{max} = \sigma_1 \geq \sigma_2 \geq \sigma_3 \ldots \geq \sigma_k = \sigma_{min}$$

We apply these ideas to $||\mathbf{G}||$. Taking $\mathbf{\Lambda} = \mathbf{G^H G}$, we have that

$$||\mathbf{G}|| = \sup_{||U|| \neq 0} \sqrt{\frac{U^H \mathbf{G^H G} U}{U^H U}} = \sqrt{\lambda_{max}\{\mathbf{G^H G}\}} = \sigma_{max}$$

Tracking

We next consider the frequency-domain conditions necessary to give good tracking of reference signals. We recall that $E(s) = \mathbf{S_o}(s)R(s)$. We can obtain a combined measure of the magnitude of the errors in all channels by considering the Euclidean form of $E(j\omega)$. Hence, consider

$$||E(j\omega)||_2 = \frac{||\mathbf{S_o}(j\omega)R(j\omega)||_2}{||R(j\omega)||_2}||R(j\omega)||_2 \leq \overline{\sigma}(\mathbf{S_o}(j\omega))||R(j\omega)||_2$$

We then see that errors are guaranteed small if $\overline{\sigma}(\mathbf{S_o}(j\omega))$ is small in the frequency band where $\|R(j\omega)\|_2$ is significant. Note that $\mathbf{S_o}(s) + \mathbf{T_o}(s) = 1$, and so

$$\overline{\sigma}(\mathbf{S_o}(j\omega)) \ll 1 \Longleftrightarrow \overline{\sigma}(\mathbf{T_o}(j\omega)) \approx \underline{\sigma}(\mathbf{T_o}(j\omega)) \approx 1$$

Using properties of singular values we have

$$(\underline{\sigma}(\mathbf{I} + \mathbf{G_o}(j\omega)\mathbf{C_o}(j\omega)))^{-1} \leq |\underline{\sigma}(\mathbf{G_o}(j\omega)\mathbf{C_o}(j\omega)) - 1|^{-1}$$

Thus, we see that errors in all channels are guaranteed small if $\underline{\sigma}(\mathbf{G_o}(j\omega)\mathbf{C_o}(j\omega))$ is made as large as possible over the frequency band where $\|R(j\omega)\|_2$ is significant.

Disturbance Compensation

We next consider disturbance rejection. For the sake of illustration, we will consider only the input-disturbance case.

For the input-disturbance case, we have

$$||E||_2 \leq \overline{\sigma}(\mathbf{S_o}(j\omega)\mathbf{G_o}(j\omega))||D_i||_2$$

Furthermore, upon application of properties of singular values we have

$$||E||_2 \leq \overline{\sigma}(\mathbf{S_o}(j\omega))\overline{\sigma}(\mathbf{G_o}(j\omega))||D_i||_2$$

We conclude:

Good input-disturbance compensation can be achieved if $\underline{\sigma}(\mathbf{G_o}(j\omega)\mathbf{C_o}(j\omega)) \gg 1$ over the frequency band where $\overline{\sigma}(\mathbf{G_o}(j\omega))\|D_i\|_2$ is significant.

Measurement-Noise Rejection

The effect of measurement noise on MIMO loop performance can also be quantified by using singular values, as shown below. We have, that, for measurement noise,

$$||Y||_2 \leq \overline{\sigma}(\mathbf{T_o}(j\omega))||D_m||_2$$

Thus, good noise rejection is achieved if $\overline{\sigma}(\mathbf{T_o}(j\omega))$ «1 over the frequency band where the noise is significant.

Directionality in Sensitivity Analysis

The preceding analysis produced upper and lower bounds that can be used as indicators of loop performance. However, the analysis presented so far has not emphasized one of the most significant features of MIMO systems, namely, directionality.

Example

In a MIMO control loop, the complementary sensitivity is given by

$$\mathbf{T_o}(s) = \begin{bmatrix} \frac{9}{s^2+5s+9} & \frac{-s}{s^2+5s+9} \\ \frac{s}{s^2+5s+9} & \frac{3(s+3)}{s^2+5s+9} \end{bmatrix}$$

The loop has output disturbances given by

$$d_o(t) = \left[K_1 \sin(\omega_d t + \alpha_1) \quad K_2 \sin(\omega_d t + \alpha_2) \right]^T$$

Determine the frequency ω_d, the ratio K_1/K_2, and the phase difference $\alpha_1 - \alpha_2$ that maximize the Euclidean norm of the stationary error, $||E||_2$.

In steady state, the error is a vector of sine waves with frequency ω_d. We then apply phasor analysis. The phasor representation of the output disturbance is

$$D_o = \left[K_1 e^{j\alpha_1} \quad K_2 e^{j\alpha_2}\right]^T$$

We see that the error due to output disturbances is the negative of that for a reference signal. Then, we have that, for every ratio K_1/K_2 such that $\|D_0\| = 1$, the following holds:

$$||E(j\omega_d)||_2 = \max_{\omega \in \mathbb{R}} ||E(j\omega)||_2 \leq \max_{\omega \in \mathbb{R}} \overline{\sigma}(\mathbf{S_o}(j\omega)) = ||\mathbf{S_o}||_\infty$$

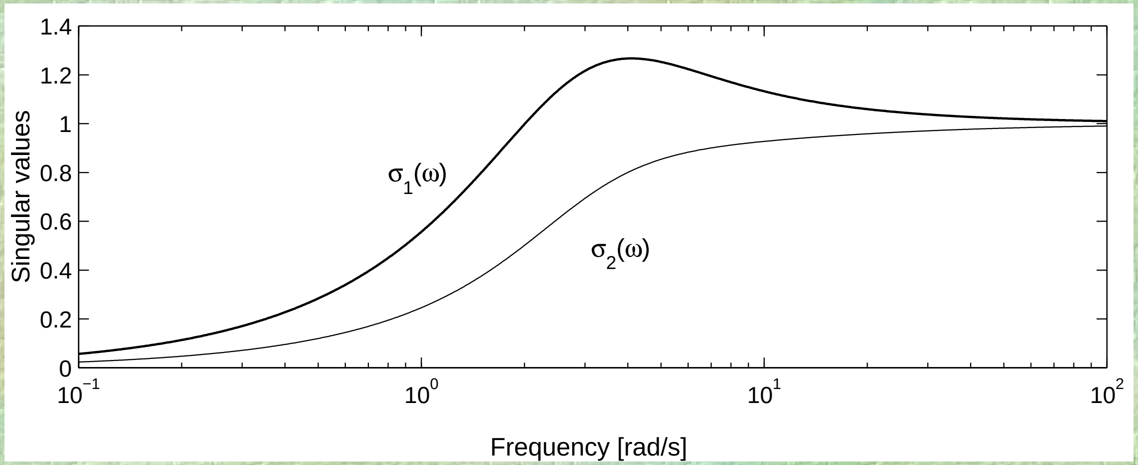
The upper bound on the above slide is reached precisely when the direction of the disturbance phasor coincides with that of the principal direction associated with the maximum singular value of $\mathbf{S_o}(j\omega_d)$.

To investigate the above result for the particular system in this example, we first obtain $\mathbf{S_o}(s)$, by applying the identity $\mathbf{T_o}(s) + \mathbf{S_o}(s) = \mathbf{I}$;

$$\mathbf{S_o}(s) = \begin{bmatrix} \frac{s(s+5)}{s^2+5s+9} & \frac{s}{s^2+5s+9} \\ \frac{-s}{s^2+5s+9} & \frac{s(s+2)}{s^2+5s+9} \end{bmatrix}$$

We can now compute the value of ω at which $\overline{\sigma}(\mathbf{S_o}(j\omega))$ is maximal. The singular values of $\mathbf{S_o}(j\omega)$ are shown on the next slide.

Figure 20.4: *Singular values of the sensitivity function*



We see that $\overline{\sigma}(\mathbf{S_o}(j\omega))$ is maximum at $\omega \approx 4.1$[rad/s]. Thus, the maximizing disturbance frequency is $\omega_d = 4.1$[rad/s].

The principal directions associated with the two singular values $\overline{\sigma}(\mathbf{S_o}(j\omega_d))$ and $\underline{\sigma}(\mathbf{S_o}(j\omega_d))$ are, respectively given by

$$u_1 = \begin{bmatrix} 0.884 + j0.322 & -0.219 + j0.260 \end{bmatrix}^T$$

$$u_2 = \begin{bmatrix} -0.340 + j0.003 & -0.315 + j0.886 \end{bmatrix}^T$$

We are interested only in u_1, which can be expressed as

$$u_1 = \begin{bmatrix} 0.944\angle 0.35 & 0.340\angle 2.27 \end{bmatrix}^T$$

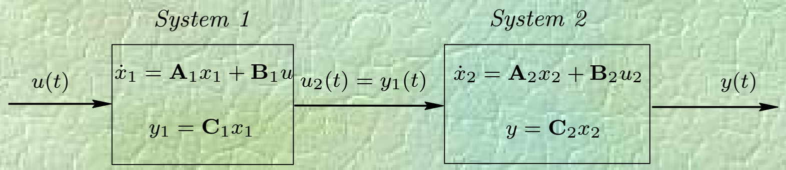
Thus, the maximizing solution (*i.e.*, *the worst case*) happens when

$$\frac{K_1}{K_2} = \frac{0.944}{0.34} = 2.774 \qquad \text{and} \qquad \alpha_1 - \alpha_2 = 0.35 - 2.27[rad] = -1.92[rad]$$

Directionality in Connection with Pole-Zero Cancellations

Directionality issues also show up in connection with pole-zero cancellations and loss of controllability or observability. Consider, for example, the set-up shown on the next slide.

Figure 17.3


System 1
System 2
$u(t)$
$\dot{x}_1 = \mathbf{A}_1 x_1 + \mathbf{B}_1 u$
$y_1 = \mathbf{C}_1 x_1$
$u_2(t) = y_1(t)$
$\dot{x}_2 = \mathbf{A}_2 x_2 + \mathbf{B}_2 u_2$
$y = \mathbf{C}_2 x_2$
$y(t)$

The composite system has realization (**A**, **B**, **C**), where

$$\mathbf{A} = \begin{bmatrix} \mathbf{A_1} & \mathbf{0} \\ \mathbf{B_2 C_1} & \mathbf{A_2} \end{bmatrix} \quad ; \mathbf{B} = \begin{bmatrix} \mathbf{B_1} \\ \mathbf{0} \end{bmatrix}$$

$$\mathbf{C} = [0 \quad \mathbf{C_2}] = 0$$

We know, from Chapter 3, that pole-zero cancellations play a role in loss of observability or controllability. However, in the MIMO case, directions are also important, as is shown in the following lemma.

Lemma 20.5: The composite system loses observability if and only if β is a pole of system 1 and a zero of system 2 such that there exist an $x_1 \in$ the null space of $(\beta \mathbf{I} - \mathbf{A}_1)$ and $\mathbf{C}_1 x_1 \in$ the null space of $\mathbf{C}_2(\beta \mathbf{I} - \mathbf{A}_2)^{-1}\mathbf{B}_2$.

Proof: See the book.

Lemma 20.6: The composite system loses controllability if and only if α is a zero of system 1 and a pole of system 2 such that there exist $x_2^T \in$ the left null space of $(\alpha \mathbf{I} - \mathbf{A}_2)$ and $x_2^T \mathbf{B}_2 \in$ left null space of $\mathbf{C}_1(\alpha \mathbf{I} - \mathbf{A}_1)^{-1}\mathbf{B}_1$.

Proof: See the book.

Example

Consider two systems, S_1 and S_2 having, respectively, the transfer functions

$$\mathbf{G_1}(s) = \begin{bmatrix} \dfrac{2s+4}{(s+1)(s+3)} & \dfrac{-2}{(s+1)(s+3)} \\ \dfrac{-3s-1}{(s+1)(s+3)} & \dfrac{-5s-7}{(s+1)(s+3)} \end{bmatrix}$$

$$\mathbf{G_2}(s) = \begin{bmatrix} \dfrac{4}{(s+1)(s+2)} & \dfrac{-1}{s+1} \\ \dfrac{2}{s+1} & \dfrac{-1}{2(s+1)(s+2)} \end{bmatrix}$$

We first build a state space representation for the system $\mathcal{S}_1$ by using the MATLAB command ***ss***.

It is straightforward (using MATLAB) command ***eig***) to compute the system eigenvalues, which are located at -1 and -3, with eigenvectors w_1 and w_2 given by

$$w_1^T = [0.8552 \quad 0.5184]; \qquad w_2^T = [-0.5184 \quad 0.8552]$$

Also, this system has no zeros.

The system S_2 has three poles, located at -1, -2 and -2 and one zero, located at -3. This zero has a left direction and a right direction *h*, which are given by

$$\mu^T = [1 \quad 2]; \qquad h^T = [1 \quad -4]$$

We observe that one pole of S_1 coincides with one zero of S_2.

S_1 output is the input of S_2:

To investigate a possible loss of observability, we have to compute $\mathbf{C}_1 w_2$ and compare it with h. We first obtain $\mathbf{C}_1 w_2 = [-1.414 \quad 5.657]^T$, from which we see that this vector is linearly dependent with h.

Thus, in this connection, there will be an unobservable mode, e^{-3t}.

S_2 output is the input of S_1:

To investigate a possible loss of controllability, we have to compute $w_2^T\mathbf{B}_1$ and compare it with μ. We have that $w_2^T\mathbf{B}_1$ = [-0.707   -0.707]. Thus, this vector is linearly independent of μ, and hence no loss of controllability occurs in this connection.

Robustness Issues

Finally, we extend the robustness results for SISO to the MIMO case. As for the SISO case, MIMO models will usually be only approximate descriptions of any real system. Thus, the performance of the nominal control loop can significantly differ from the true or achieved performance. To gain some insight into this problem, we consider linear modeling errors, as we did for SISO systems.

We will consider two equivalent forms for multiplicative modeling errors (MME):

$$\mathbf{G}(s) = (\mathbf{I} + \mathbf{G}_{\Delta \mathbf{l}}(s))\mathbf{G}_{\mathbf{o}}(s) = \mathbf{G}_{\mathbf{o}}(s)(\mathbf{I} + \mathbf{G}_{\Delta \mathbf{r}}(s))$$

where $\mathbf{G}_{\Delta 1}(s)$ and $\mathbf{G}_{\Delta r}(s)$ are the left and right MME matrices, respectively. We observe that these matrices are related by

$$\mathbf{G}_{\Delta \mathbf{l}}(s) = \mathbf{G}_{\mathbf{o}}(s)\mathbf{G}_{\Delta \mathbf{r}}(s)[\mathbf{G}_{\mathbf{o}}(s)]^{-1}; \qquad \mathbf{G}_{\Delta \mathbf{r}}(s) = [\mathbf{G}_{\mathbf{o}}(s)]^{-1}\mathbf{G}_{\Delta \mathbf{l}}(s)\mathbf{G}_{\mathbf{o}}(s)$$

This equivalence allows us to derive expressions by using either one of the descriptions. For simplicity, we will choose the left MME matrix and will examine the two main sensitivities only: the sensitivity and the complementary sensitivity.

We can then derive expressions for the achievable sensitivities

$$\mathbf{S}(s) = [\mathbf{I} + \mathbf{G}(s)\mathbf{C}(s)]^{-1} = [\mathbf{I} + \mathbf{G_o}(s)\mathbf{C}(s) + \mathbf{G_{\Delta l}}(s)\mathbf{G_o}(s)\mathbf{C}(s)]^{-1}$$

$$= [\mathbf{I} + \mathbf{G_o}(s)\mathbf{C}(s)]^{-1}[\mathbf{I} + \mathbf{G_{\Delta l}}(s)\mathbf{T_o}(s)]^{-1} = \mathbf{S_o}(s)[\mathbf{I} + \mathbf{G_{\Delta l}}(s)\mathbf{T_o}(s)]^{-1}$$

$$\mathbf{T}(s) = \mathbf{G}(s)\mathbf{C}(s)[\mathbf{I} + \mathbf{G}(s)\mathbf{C}(s)]^{-1} = [\mathbf{I} + \mathbf{G_{\Delta l}}(s)]\mathbf{T_o}(s)[\mathbf{I} + \mathbf{G_{\Delta l}}(s)\mathbf{T_o}(s)]^{-1}$$

Note the similarity between the above expressions and those for the SISO case. We can also use these expressions to obtain robustness results.

Theorem 20.2: Consider a plant with nominal and true transfer function $\mathbf{G_o}(s)$ and $\mathbf{G}(s)$, respectively. Assume that they are related by

$$\mathbf{G}(s) = (\mathbf{I} + \mathbf{G_{\Delta l}}(s))\mathbf{G_o}(s) = \mathbf{G_o}(s)(\mathbf{I} + \mathbf{G_{\Delta r}}(s))$$

Also assume that a controller $\mathbf{C}(s)$ achieves nominal internal stability and that $\mathbf{G_o}(s)\mathbf{C}(s)$ and $\mathbf{G}(s)\mathbf{C}(s)$ have the same number, P, of unstable poles. Then a sufficient condition for stability of the feedback loop obtained by applying the controller to the true plant is

$$\overline{\sigma}(\mathbf{G_{\Delta l}}(j\omega)\mathbf{T_o}(j\omega)) < 1 \qquad \forall \omega \in \mathsf{R}$$

Proof: See the book.

Example 20.8

A MIMO plant has nominal and true models given by $\mathbf{G_o}(s)$ and $\mathbf{G}(s)$, respectively, where

$$\mathbf{G_o}(s) = \begin{bmatrix} \dfrac{2}{s+1} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{1}{(s+1)(s+2)} & \dfrac{2}{s+2} \end{bmatrix}$$

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{20}{(s+1)(s+10)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{1}{(s+1)(s+2)} & \dfrac{40}{(s+2)(s+20)} \end{bmatrix}$$

We see that $\mathbf{G}_{\Delta 1}(s)$, can be computed from

$$\mathbf{G}_{\boldsymbol{\Delta}\mathbf{l}}(s) = \mathbf{G}(s)[\mathbf{G}_{\mathbf{o}}(s)]^{-1} - \mathbf{I}$$

This yields

$$\mathbf{G}_{\boldsymbol{\Delta}\mathbf{l}}(s) = \begin{bmatrix} \dfrac{-4s^3 - 12s^2 - 8s}{4s^3 + 52s^2 + 127s + 70} & \dfrac{2s^2 + 4s}{4s^3 + 52s^2 + 127s + 70} \\ \dfrac{2s^2 + 2s}{4s^3 + 92s^2 + 247s + 140} & \dfrac{-4s^3 - 12s^2 + 8s}{4s^3 + 92s^2 + 247s + 140} \end{bmatrix}$$

The singular values of $\mathbf{G}_{\Delta 1}(s)$ are computed by using MATLAB commands, leading to the plots shown below.

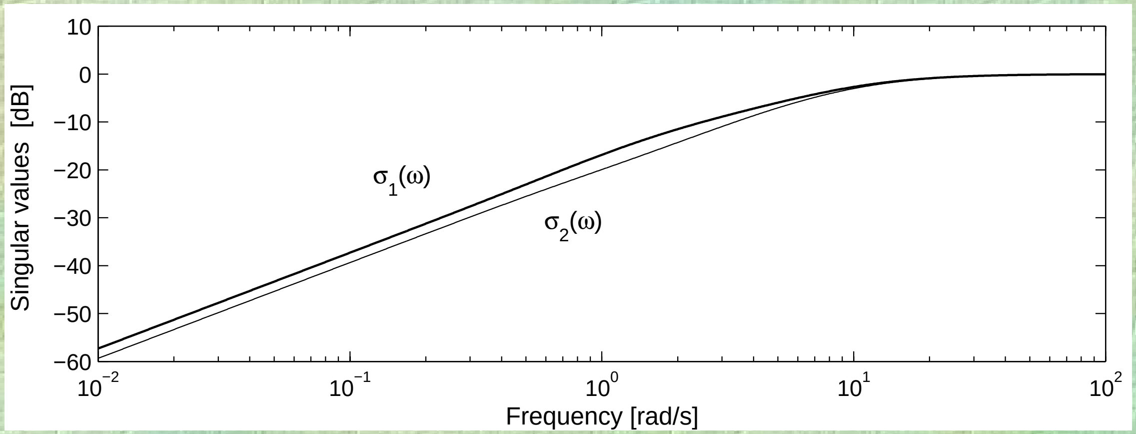


Figure 20.5: *Singular values of MME matrix*

We can use the above data to determine what values of $T_0(s)$ give guaranteed closed loop stability.

Summary

- In previous chapters, we have considered the problem of controlling a single output by manipulating a single input (SISO).
- Many control problems, however, require that multiple outputs be controlled simultaneously; to do so, multiple inputs much be manipulated - usually, subtly orchestrated (MIMO).
  - Aircraft autopilot example: speed, altitude, pitch, roll, and yaw angles must be maintained; throttle, several rudders, and flaps are available as control variables.
  - Chemical process example: yield and throughput must be regulated; thermal energy, valve actuators, and various utilities are available as control variables.

- The key difficulty in achieving the necessary orchestration of inputs is the *multivariable interactions*, also known as *coupling*.
- From an input-output point of view, two fundamental phenomena arise from coupling - See Figure 20.6.

Figure 20.6: *Two phenomena associated with multivariable interactions*

- Multivariable interactions in the form, shown in Figure 20.6 add substantial complexity to MIMO control.
- Both state space and transfer-function models can be generalized to MIMO models.
- The MIMO transfer-function matrix can be obtained from a state space model by $\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$.
- In general, if the model has m inputs, $u \in \mathbb{R}^m$, and 1 outputs, $y \in \mathbb{R}^\ell$, then
  - the transfer-function matrix consists of an $l \times m$ matrix of SISO transfer functions, and
  - for an n-dimensional state vector, $x \in \mathbb{R}^n$, the state space model matrices have dimensions $\mathbf{A} \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$, $D \in \mathbb{R}^{l\times m}$.

- Some MIMO model properties and analysis results generalize quite straightforwardly from SISO theory:
  - similarity transformations among state space realizations
  - observability and controllability
  - poles
- Other MIMO properties are more subtle or complex than their SISO counterparts, usually due to interactions or the fact that matrices do not commute - e.g.,
  - zeros
  - left and right matrix fractions.