Chapter 12 - Solved Problems

Solved Problem 12.1. *The Z-transform of a signal* $f[k]$ *is given by*

$$F_q(z) = \frac{2z^6 - z^5 + 3z^3 + 2z^2}{z^7 + 2z^6 + z^5 + z^4 + 0.5} = \frac{n(z)}{d(z)} \tag{1}$$

where $n(z)$ *and* $d(z)$ *are the numerator and denominator polynomials respectively*
Compute $f[3]$.

Solutions to Solved Problem 12.1

Solved Problem 12.2. *Assume that the response of a discrete time system to a Kronecker delta (with zero initial conditions) is given by*

$$h[k] = 2(0.5)^k - 2(0.2)^k \tag{2}$$

12.2.1 *Find the system transfer function.*

12.2.2 *Find the system recursive equation in shift operator form.*

Solutions to Solved Problem 12.2

Solved Problem 12.3. *A discrete time system, with input* $u[k]$ *and output* $y[k]$*, has a transfer function given by*

$$G_q(z) = \frac{z - 0.8}{z^2 - 1.3z + 0.42} \tag{3}$$

Compute the unit step response with zero initial conditions.

Solutions to Solved Problem 12.3

Solved Problem 12.4. *The transfer function of a discrete time system is given by*

$$G_q(z) = \frac{0.5z}{(z + 0.5)(z - 0.5)} \tag{4}$$

Compute, if it exists, the steady state response of the system to a unit constant input ($\forall k \geq 0$) *and initial conditions* $y[-1] = -1$, $y[-2] = 3$.

Solutions to Solved Problem 12.4

Solved Problem 12.5. *A signal* $f(t) = 2 - 2\cos(2\pi t)$ *is sampled every* Δ $[s]$.
Compute the Z-transform of the sampled sequence $f[k]$ *for* $\Delta = 0.1$ $[s]$.

Solutions to Solved Problem 12.5

Solved Problem 12.6. *Consider a continuous time transfer function*

$$G_o(s) = 3\frac{-s + 1}{(s + 1)(s + 3)} \tag{5}$$

Compute the associated pulse-transfer function, $H_{oq}(z)$*, assuming that the sampling period is* $\Delta = 0.1$ $[s]$ *and that a zero order hold is employed.*

Solutions to Solved Problem 12.6

Solved Problem 12.7. *A discrete time system has a transfer function given by*

$$G_q(z) = \frac{0.1(z-0.2)}{(z-0.8)(z-0.9)} \tag{6}$$

Assume that the input is a sine wave of the form $u[k] = 2\cos(0.2\pi k)$. *Compute the steady state output,* $y[k]$, *if it exists.*

Solutions to Solved Problem 12.7

Chapter 12 - Solutions to Solved Problems

Solution 12.1. *To solve this problem we could compute the analytical expression for the inverse Z-transform, and then we could evaluate that expression at $k = 3$. An alternative method is to recall that*

$$F_q(z) = f[0] + f[1]z^{-1} + f[2]z^{-2} + f[3]z^{-3} + f[4]z^{-4} + \dots \tag{7}$$

i.e., $f[k]$ can be computed by expanding the fraction in (1) in powers of z^{-1}. This can be done by dividing $n(z)$ by $d(z)$ up to the term z^{-3}, its coefficient is equal to $f[3]$.

*The division can be performed using the MATLAB command **deconv**. This requires us to perform the division of $z^3 n(z)$ by $d(z)$. If we do that we obtain*

$$F_q(z) = 2z^{-1} - 5z^{-2} + 8z^{-3} + \dots \tag{8}$$

Therefore $f[3] = 8$.

Solution 12.2.

12.2.1 *The transfer function is the Z-transform of the system response to a Kronecker delta (with zero initial conditions). Hence (use Table 12.1 in the book.)*

$$H_q(z) = \mathcal{Z}\left[h[k]\right] = \frac{2z}{z-0.5} - \frac{2z}{z-0.2} = \frac{0.6z}{(z-0.5)(z-0.2)} \tag{9}$$

This result can also be obtained using a symbolic mathematical software package, such as MAPLE. In this case, the MAPLE code would be

```
>h(k):=2*(1/2)^ k-2*(1/5)^k;
>H(z):=simplify(ztrans(h(k),k,z));
```

12.2.2 *Assuming that the system input is $u[k]$ and the system output as $y[k]$, with Z-transforms $U_q(z)$ and $Y_q(z)$, respectively, then*

$$\frac{Y_q(z)}{U_q(z)} = H_q(z) \Longrightarrow z^2 Y_q(z) - 0.7zY_q(z) + 0.1Y_q(z) = 0.6zU_q(z) \tag{10}$$

$$\Longrightarrow Y_q(z) - 0.7z^{-1}Y_q(z) + 0.1z^{-2}Y_q(z) = 0.6z^{-1}U_q(z) \tag{11}$$

$$\Longrightarrow \boxed{\text{y[k]-0.7y[k-1]+0.1y[k-2]=0.6u[k-1]}} \tag{12}$$

Solution 12.3. *The Z-transform of the input is given by*

$$U_q(z) = \frac{z}{z-1} \tag{13}$$

Then the Z-transform of the output is given by

$$Y_q(z) = z \underbrace{\frac{z-0.8}{(z-0.6)(z-0.7)(z-1)}}_{\tilde{Y}_q(z)} \tag{14}$$

If we expand $\tilde{Y}_q(z)$ in partial fractions[1] *we have*

$$Y_q(z) = z\left[\frac{5}{3(z-1)} - \frac{5}{z-0.6} + \frac{10}{3(z-0.7)}\right] = \frac{5z}{3(z-1)} - \frac{5z}{z-0.6} + \frac{10z}{3(z-0.7)} \tag{15}$$

Applying the inverse Z-transform we finally obtain

$$\boxed{y[k] = \frac{5}{3}\mu[k] - 5(0.6)^k + \frac{10}{3}(0.7)^k; \quad \forall k \geq 0} \tag{16}$$

Note the trick used in equation (15), where the whole expression has been factorized by z. This facilitates the computation of the inverse transform, otherwise a delay has to be introduced in the time function.

The step response of a linear system can also be computed using a software package similar to MAPLE, using code similar to

```
>Gq(z):=(z-0.8)/((z-0.6)*(z-0.7));
>invztrans(Gq(z)*z/(z-1),z,k);
```

Solution 12.4. *The system has natural frequencies at 0.5 and −0.5, since both have magnitude less than one, then the system is stable. The system stability in conjunction with the nature of the input (a constant) yields an output which converges to a constant.*

Note that, in this case, the initial conditions have no effect whatsoever on the steady state behavior, since they only modify the initial amplitude of the natural modes, which vanish as time progresses.

To compute the steady state value, we use the final value theorem (see Table 12.2 in the book), to obtain

$$y[\infty] = \lim_{z \to 1}(z-1)Y_q(z) = \lim_{z\to 1}(z-1)\left[G_q(z)\frac{z}{z-1}\right] = G_q(1) = \frac{2}{3} \tag{17}$$

Note that $G_q(1)$ is the d.c. gain of the system.

Solution 12.5. *The sampled signal is given by*

$$f[k] = 2 - 2\cos(0.2\pi k) \tag{18}$$

Using Table 12.1 in the book we have that

$$F_q(z) = \frac{2z}{z-1} - \frac{2z(z-\cos(0.2\pi))}{z^2 - 2z\cos(0.2\pi) + 1} = \frac{0.38z(z+1.0)}{(z-1)(z^2 - 1.618z + 1)} \tag{19}$$

---

[1] For this you can use the MATLAB command **residue**.

Solution 12.6. *We first compute the unit step response of $G_o(s)$, this is given by*

$$g(t) = \mathcal{L}^{-1}\left[\frac{G_o(s)}{s}\right] = \mathcal{L}^{-1}\left[\frac{3(-s+1)}{(s+1)(s+3)s}\right] \tag{20}$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{3}{s+1} + \frac{2}{s+3}\right] = 1 - 3e^{-t} + 2e^{-3t} \tag{21}$$

To apply (12.13.4) from the book, we next need to compute the Z-transform of the sequence $g[k] \triangleq g(k\Delta)$:

$$\mathcal{Z}\left[g[k]\right] = \mathcal{Z}\left[1 - 3(e^{-\Delta}) + 2(e^{-3\Delta})\right] \tag{22}$$

$$= \frac{z}{z-1} - \frac{3z}{z-e^{-\Delta}} + \frac{2z}{z-e^{-3\Delta}} = \frac{z}{z-1} - \frac{3z}{z-0.9048} + \frac{2z}{z-0.7408} \tag{23}$$

$$= z\frac{-0.2328\,z + 0.2575}{(z-1)(z-0.9048)(z-0.7408)} \tag{24}$$

We are finally in position to apply (12.13.4) from the book. This yields

$$H_{oq}(z) = (1-z^{-1})\mathcal{Z}\left[g[k]\right] = \frac{-0.2328\,z + 0.2575}{(z-0.9048)(z-0.7408)} \tag{25}$$

The above computation (very painful for a more complicated $G_o(s)$) can be done using the MATLAB command ***c2d****, as shown in the following code*

```
>>Go=tf([-3 3],[1 4 3]);
>>Hoq=c2d(Go,0.1,'zoh')
```

Solution 12.7. *We observe that the system is stable (its poles are located inside the unit disk). Hence the steady state output can be computed using the frequency response concepts explained in section §12.15 of the book.*

Thus, the key quantity is $G_q(e^{j\theta})$ where, in this example, $\theta = 0.2\pi$. This can be computed using the MATLAB command ***freqresp****.*

$$G_q(e^{j0.2\pi}) = 0.242e^{-j2.51} \Longrightarrow y[k] = 0.484\cos(0.2\pi k - 2.51); \quad \textit{in steady state} \tag{26}$$
