Chapter 17 - Solved Problems

Solved Problem 17.1. *A discrete time system, with input $u[k]$ and output $y[k]$, has a transfer function given by*

$$G_q(z) = \frac{z - 0.8}{z^2 - 1.3z + 0.42} \tag{1}$$

Build a state space representation.

Solutions to Solved Problem 17.1

Solved Problem 17.2. *A continuous time system, with input $u(t)$ and output $y(t)$, has a transfer function given by*

$$G(s) = \frac{s + 4}{(s + 5)(s + 1)} \tag{2}$$

Assume that the **A** *matrix in a state space description is chosen to be*

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix} \tag{3}$$

Find, if they exist, a pair of matrices **B** *and* **C** *such that the 4-tuple (***A***,* **B***,* **C***,0) describes the system.*

Solutions to Solved Problem 17.2

Solved Problem 17.3. *A linear continuous time system is described in state space form with the following matrices*

$$\mathbf{A} = \begin{bmatrix} -2 & \epsilon \\ 4 & -5 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}; \quad \mathbf{C} = [1 \ \ -1]; \quad \mathbf{D} = 0 \tag{4}$$

17.3.1 *Show that the system is completely controllable if and only if $\epsilon \neq 0$.*

17.3.2 *Compute the transfer function, $G(s)$, from $U(s)$ to $Y(s)$, and show that there is a pole-zero cancellation if $\epsilon = 0$ (sufficiency). Is it a necessary condition?*

Solutions to Solved Problem 17.3

Solved Problem 17.4. *Consider the mechanical system shown in Figure 1.*

An external force $f(t)$ is applied to the system. Assume also that the movement of blocks 1 and 2 is opposed by viscous friction forces $f_{D1}(t)$ and $f_{D2}(t)$ respectively. These forces are given by

$$f_{D1}(t) = D_1 v_1(t); \quad f_{D2}(t) = D_2 v(t) \tag{5}$$

17.4.1 *Build a state space representation for the system*

17.4.2 *Find an expression for the energy stored in the system at time t. Express this energy as a function of the state.*

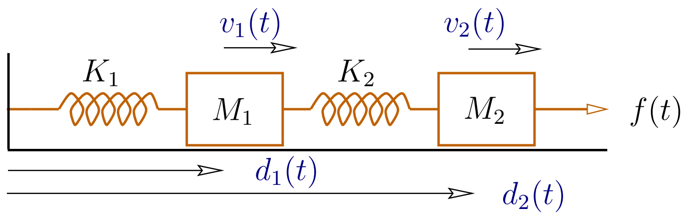

Figure 1: Mechanical system

17.4.3 *Compute the system natural frequencies and plot them on the complex plane for the following set of parameters.*

Set 1 $M_1 = 2\ [kg]$, $M_2 = 5\ [kg]$, $K_1 = 30\ [N/m]$, $K_2 = 50\ [N/m]$, $D_1 = 10\ [Ns/m]$, $D_2 = 15\ [Ns/m]$

Set 2 $M_1 = 2\ [kg]$, $M_2 = 5\ [kg]$, $K_1 = 30\ [N/m]$, $K_2 = 50\ [N/m]$, $D_1 = 2\ [Ns/m]$, $D_2 = 3\ [Ns/m]$

Discuss

Solutions to Solved Problem 17.4

Solved Problem 17.5. *Consider the mechanical system described in Solved Problem 17.4. Assume that the system parameters are those belonging to Set 2, and that the system output, $y(t)$ is the velocity of the second block.*

A natural choice of state variables is

$$\mathbf{x(t)} = \begin{bmatrix} d_1(t) & d_2(t) & v_1(t) & v_2(t) \end{bmatrix}^T \tag{6}$$

This choice leads to the state space model defined by the 4-tuple $(\mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$.
Assume that we decide instead to choose the state as

$$\overline{\mathbf{x}}(\mathbf{t}) = \begin{bmatrix} d_1(t) & d_2(t) - d_1(t) & v_1(t) & v_2(t) - v_1(t) \end{bmatrix}^T \tag{7}$$

This choice leads to the state space model defined by the 4-tuple $(\overline{\mathbf{A}}, \overline{\mathbf{B}}, \overline{\mathbf{C}}, 0)$.
Find $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$ *and* $\overline{\mathbf{C}}$.

Solutions to Solved Problem 17.5

Chapter 17 - Solutions to Solved Problems

Solution 17.1. *For this problem we will use the results presented in section §17.5 of the book. We then have that*

$$Y_q(z) = \underbrace{\frac{z}{z^2 - 1.3z + 0.42} U_q(z)}_{X_2(z)} - 0.8 \underbrace{\frac{1}{z^2 - 1.3z + 0.42} U_q(z)}_{X_1(z)} \tag{8}$$

$$y[k] = x_2[k] - 0.8 x_1[k] \tag{9}$$

Hence $X_2(z) = zX_1(z)$. *Therefore*

$$x_1[k+1] = x_2[k] \tag{10}$$

On the other hand

$$U_q(z) = \frac{z^2 - 1.3z + 0.42}{z^2 - 1.3z + 0.42} U_q(z) \tag{11}$$

$$= \frac{z^2}{z^2 - 1.3z + 0.42} U_q(z) - 1.3 \frac{z}{z^2 - 1.3z + 0.42} U_q(z) + 0.42 \frac{1}{z^2 - 1.3z + 0.42} U_q(z) \tag{12}$$

$$= zX_2(z) - 1.3X_2(z) + 0.42X_1(z) \tag{13}$$

$$\Longrightarrow u[k] = x_2[k+1] - 1.3x_2[k] + 0.42x_1[k] \tag{14}$$

Then, combining (9), (10) *and* (14), *we obtain*

$$\begin{bmatrix} x_1[k+1] \\ x_2[k+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.42 & 1.3 \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[k] \tag{15}$$

$$y[k] = [1 \ \ -0.8] \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} \tag{16}$$

Solution 17.2. *We first verify that the eigenvalues of* $\mathbf{A}$ *are equal to the transfer function poles. This is true since the diagonal matrix* $\mathbf{A}$ *has eigenvalues at* -1 *and* -5, *which are also the locations of the poles.*

We next describe $\mathbf{B}$ *and* $\mathbf{C}$ *as follows*

$$\mathbf{B} = \begin{bmatrix} b_1 & b_2 \end{bmatrix}^T; \quad \mathbf{C} = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \tag{17}$$

Then

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ 0 & s+5 \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{18}$$

$$= \frac{c_1 b_1}{s+1} + \frac{c_2 b_2}{s+5} = \frac{(c_1 b_1 + c_2 b_2)s + 5c_1 b_1 + c_2 b_2}{(s+1)(s+5)} \tag{19}$$

Comparing (2) and (19) we have that

$$1 = c_1 b_1 + c_2 b_2 \tag{20}$$
$$4 = 5c_1 b_1 + c_2 b_2 \tag{21}$$

From where $c_1 b_1 = 0.75$ *and* $c_2 b_2 = 0.25$. *As expected these results say that there are an infinite number of possible choices for* $\mathbf{B}$ *and* $\mathbf{C}$. *For instance we can choose* $c_1 = 0.25$, $b_1 = 3$, $c_2 = 0.25$ *and* $b_2 = 1$. *To verify this we can use the following MATLAB code.*

```
>>A=[-1 0;0 -5];B=[3;1];C=[0.25 0.25];
>>[numG,denG]=ss2tf(A,B,C,0)
```

Solution 17.3.

17.3.1 *We first compute the controllability matrix* $\mathbf{\Gamma}_c$ *defined in equation (17.6.14) of the book:*

$$\mathbf{\Gamma}_c = [\mathbf{B}\ \mathbf{AB}] = \begin{bmatrix} 0 & 3\epsilon \\ 3 & -5\epsilon \end{bmatrix} \tag{22}$$

Since the system is completely controllable ***if and only if*** $\mathbf{\Gamma}_c$ *has maximum rank, then the system is completely controllable if and only if* $\epsilon \neq 0$.

17.3.2 *The transfer function is given by*

$$G(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} = [1\ \ -1] \begin{bmatrix} s+2 & -\epsilon \\ -4 & s+5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 3 \end{bmatrix} \tag{23}$$
$$= -3 \frac{(s+2) - \epsilon}{(s+5)(s+2) - 4\epsilon} \tag{24}$$

The system zero is located at $s = c = -2 + \epsilon$. *Cancellation occurs if and only if this is also a pole of the system. To analyze that condition we substitute* $s = c$ *in the denominator. The result is* $\epsilon(1 - \epsilon)$. *This proves that* $\epsilon = 0$ *is sufficient to have a pole-zero cancellation. However we also see that the same happens if* $\epsilon = 1$. *For this value of* ϵ, *the system is completely controllable (*$\mathbf{\Gamma}_c$ *is full rank). Notwithstanding this, the system is not completely observable as it can be verified computing the observability matrix,* $\mathbf{\Gamma}_o$, *as defined in (17.7.3)*[1]

$$\mathbf{\Gamma}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -6 & 6 \end{bmatrix} \tag{25}$$

which has rank 1.

We conclude that $\epsilon = 0$ *is sufficient, but no necessary for the system transfer function to have a pole-zero cancellation, since* $\epsilon = 1$ *also yields a cancellation. The first case corresponds to a deficiency in controllability, while the second case corresponds to a lack of observability.*

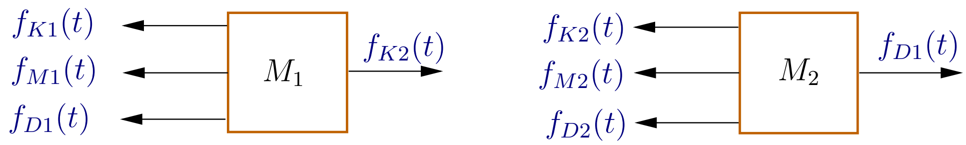

Figure 2: Mechanical system. Free body equilibrium

Solution 17.4. *We consider the free body dynamic equilibrium for every block, as shown in Figure 2. where*

$$f_{K1}(t) = K_1 d_1(t); \qquad f_{K2}(t) = K_2(d_2(t) - d_1(t)) \tag{26}$$

$$f_{M1}(t) = M_1 \frac{dv_1(t)}{dt}; \qquad f_{M2}(t) = M_2 \frac{dv_2(t)}{dt} \tag{27}$$

$$f_{D1}(t) = D_1 v_1(t); \qquad f_{D2}(t) = D_2 v_2(t) \tag{28}$$

17.4.1 *We first realize that the behavior of the system for $t \geq 0$ is determined by $f(t)$, $\forall t \geq 0$, and also for the energy initially stored in the two masses and in the two springs. We observe that since we can set independent initial conditions in these (and only in these) four mechanical components, the minimal dimension of the state vector is equal to 4. We then choose the states in such a way that each of them describe these four independent degrees of freedom. Say we choose*

$$x_1(t) = d_1(t); \quad x_1(t) = d_2(t); \quad x_3(t) = v_1(t); \quad x_4(t) = v_4(t); \tag{29}$$

Then, the state vector is given by

$$\mathbf{x(t)} = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) \end{bmatrix}^T \tag{30}$$

Applying Newton's laws to every free body we have that

$$f_{K2}(t) = f_{M1}(t) + f_{K1}(t) + f_{D1}(t) \tag{31}$$

$$\Longrightarrow K_2(d_2(t) - d_1(t)) = M_1 \frac{dv_1(t)}{dt} + K_1 d_1(t) + D_1 v_1(t) \tag{32}$$

$$f(t) = f_{K2}(t) + f_{M2}(t) + f_{D2}(t) \tag{33}$$

$$\Longrightarrow f(t) = K_2(d_2(t) - d_1(t)) + M_2 \frac{dv_2(t)}{dt} + D_2 v_2(t) \tag{34}$$

Using the choice of states described in equation (29), we have that

[1] Note that this matrix can be computed using the MATLAB command **obsv**.

$$\underbrace{\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \\ \frac{dx_3(t)}{dt} \\ \frac{dx_4(t)}{dt} \end{bmatrix}}_{\dot{\mathbf{x}}(\mathbf{t})} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-K_1 - K_2}{M_1} & \frac{K_2}{M_1} & -\frac{D_1}{M_1} & 0 \\ \frac{K_2}{M_2} & -\frac{K_2}{M_2} & 0 & -\frac{D_2}{M_2} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}}_{\mathbf{x}(\mathbf{t})} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{B}} \underbrace{f(t)}_{u(t)} \tag{35}$$

17.4.2 *The energy $W(t)$ stored in the system at time t, is the sum of the energies stored in each of the masses and springs, i.e.,*

$$W(t) = \frac{K_1 d_1(t)^2}{2} + \frac{M_1 v_1(t)^2}{2} + \frac{K_2(d_2(t) - d_1(t))^2}{2} + \frac{M_2 v_2(t)^2}{2} \tag{36}$$

This leads to

$$W(t) = \frac{1}{2} \begin{bmatrix} x_1(t) & x_2(t) - x_1(t) & x_3(t) & x_4(t) \end{bmatrix}^T \begin{bmatrix} K_1 & 0 & 0 & 0 \\ 0 & K_2 & 0 & 0 \\ 0 & 0 & M_1 & 0 \\ 0 & 0 & 0 & M_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) - x_1(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} \tag{37}$$

We next notice that

$$\begin{bmatrix} x_1(t) \\ x_2(t) - x_1(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} \tag{38}$$

When this relation is substituted into (37) we finally obtain

$$W(t) = \frac{1}{2} \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) \end{bmatrix}^T \begin{bmatrix} 2K_1 & -K_2 & 0 & 0 \\ -K_1 & K_2 & 0 & 0 \\ 0 & 0 & M_1 & 0 \\ 0 & 0 & 0 & M_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} \tag{39}$$

The expression above has the general structure $\mathbf{x(t)}^T \mathbf{\Lambda} \mathbf{x(t)}$, where $\mathbf{\Lambda}$ is a matrix with nonnegative eigenvalues. This is the general form of an ***energy function****.*

17.4.3 *The natural frequencies of the system correspond to the eigenvalues of matrix* $\mathbf{A}$. *To compute the eigenvalues for any set of data we use the following MATLAB code.*

```
>>K1=input('spring 1 constant K1:  ');
>>K2=input('spring 2 constant K2:  ');
>>M1=input('block 1 mass M1:  ');
>>M2=input('block 2 mass M2:  ');
>>D1=input('friction 1 constant D1:  ');
>>D2=input('friction 2 constant D2:  ');
>>A=[0 0 1 0;0 0 0 1;(-K1-K2)/M1 K2/M1 -D1/M1 0;K2/M2 -K2/M2 0 -D2/M2];
>>lambda=eig(A)
```

Using this code, we obtain the following eigenvalues of the $\mathbf{A}$*-matrix*

$$\text{Set 1} \quad -2.32 \pm j6.39 \quad \text{and} \quad -1.68 \pm j0.66 \tag{40}$$

$$\text{Set 2} \quad -0.47 \pm j6.82 \quad \text{and} \quad -0.33 \pm j1.76 \tag{41}$$

We comment as follows:

- *All the eigenvalues have negative real parts, this is not surprising since the system is stable.*
- *The eigenvalues have nonzero imaginary parts. This indicates that the natural modes of the system exhibit (damped) oscillatory behavior. This is possible since, in this system, the energy appears in two different forms: kinetic (in the masses) and potential (in the springs). However the oscillations disappear for sufficiently high damping, when only decaying exponentials are observed (try, for instance,* $D_1 = 100$ *and* $D_2 = 200$*).*
- *The natural modes for the Set 1 extinguish faster than in the case of the Set 2. This can be appreciated from Figure 3, where in the first case the eigenvalues are much farther to the left. The physical reason for this is that in the first case the damping due to friction is larger. Indeed, when the friction constants are zero, the system will have undamped natural frequencies on the imaginary axis.*

The natural frequencies for both sets of parameters are as shown in Figure 3

Solution 17.5. *We first note that the similarity transformation (see section §17.3 of the book) is determined by*

$$\overline{\mathbf{x}}(\mathbf{t}) = \mathbf{M}\mathbf{x}(\mathbf{t}); \quad \text{where } \mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \Longrightarrow \mathbf{M}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{42}$$

In the solution to Problem 17.4, evaluated for the Set 2, we obtained

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -40 & 25 & -1 & 0 \\ 10 & -10 & 0 & -0.6 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \tag{43}$$

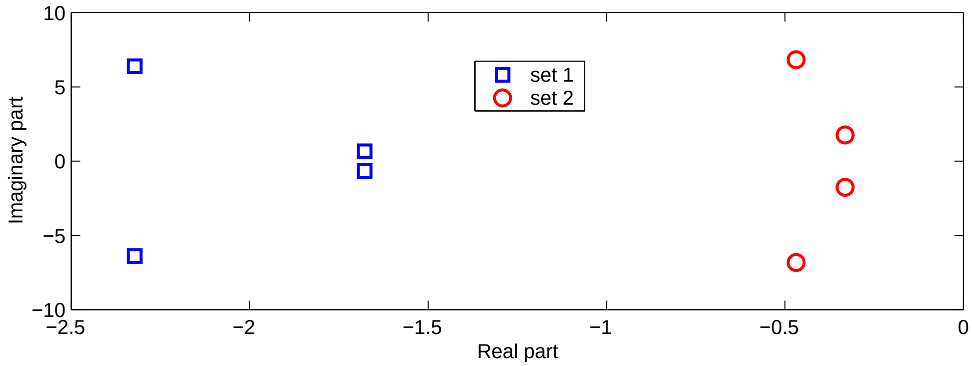

Figure 3: Natural frequencies for two sets of parameters

We can now use the relations in equation (17.3.4) of the book [2]*. This yields*

$$\overline{\mathbf{A}} = \mathbf{MAM}^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -15 & 25 & -1 & 0 \\ 15 & -35 & 0.4 & -0.6 \end{bmatrix}; \quad \overline{\mathbf{B}} = \mathbf{MB} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \quad \overline{\mathbf{C}} = \mathbf{CM}^{-1} = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} \qquad (44)$$

[2] Note that $\mathbf{M} = \mathbf{T}^{-1}$.