Chapter 3 - Solved Problems

Solved Problem 3.1. *A nonlinear system has an input-output model given by*

$$\frac{dy(t)}{dt} + (1 + 0.2y(t))y(t) = u(t) + 0.2u(t)^3 \tag{1}$$

3.1.1 *Compute the operating point(s) for $u_Q = 2$. (assume it is an equilibrium point)*

3.1.2 *Obtain a linearized model for each of the operating points above.*

Solutions to Solved Problem 3.1

Solved Problem 3.2. *A nonlinear system is described in state space form by the model*

$$\dot{x}_1(t) = -x_1(t)^2 + x_2(t) + 3u(t) \tag{2}$$
$$\dot{x}_2(t) = -2x_1(t)x_2(t) \tag{3}$$
$$y(t) = x_1(t) \tag{4}$$

Obtain a linearized model around the equilibrium point $(u_Q, y_Q) = (2, 0)$.

Solutions to Solved Problem 3.2

Solved Problem 3.3. *Consider a discrete time system with input $u[k]$ and output $y[k]$, having an input-output model given by*

$$y[k] + 0.4y[k-1] = u[k-2] \tag{5}$$

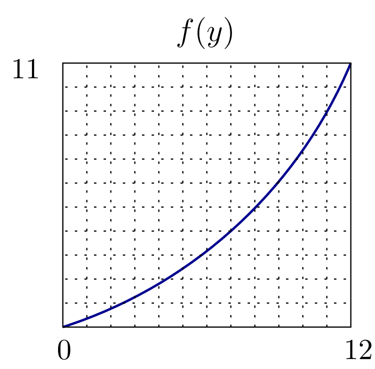
Choose state variables and build a state space model

Solutions to Solved Problem 3.3

Solved Problem 3.4. *The input-output model for a nonlinear system is given by*

$$\frac{dy(t)}{dt} + f(y) = 2u(t) \tag{6}$$

where $f(y)$ is the nonlinear function appearing in the figure.
Build a linearized model for the equilibrium point determined by $u_Q = 3$.



Solutions to Solved Problem 3.4

Solved Problem 3.5. *Consider the electric network shown in Figure 1*

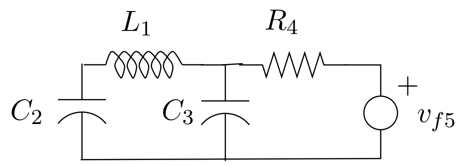


Figure 1: Electric network

3.5.1 *Without using any equations, discuss how many states the system has.*

3.5.2 *Build a state space model.*

Solutions to Solved Problem 3.5

Solved Problem 3.6. *Consider a single tank of constant cross-sectional area A. The flow of water from the tank is governed by the relationship*

$$f_{out} = K\sqrt{h} \tag{7}$$

where h is the height of liquid in the tank and K is a constant.

Assume that the flow of liquid into the tank is a control variable, u.

3.6.1 *Write down the equation governing the height of liquid in the tank.*

3.6.2 *Linearize the model about a nominal height of $h = h^*$.*

3.6.3 *Repeat part (i) and (ii) for a tank where the cross sectional area increases with height i.e., $A = ch$.*

Solutions to Solved Problem 3.6

Solved Problem 3.7. *Consider a ball in a frictionless cone which is being rotated as shown in Figure 2. Write down the equations of motion of the ball in the vertical plane.*

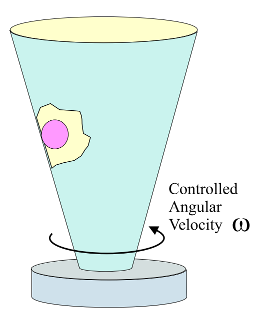


Figure 2: Cone

Solutions to Solved Problem 3.7

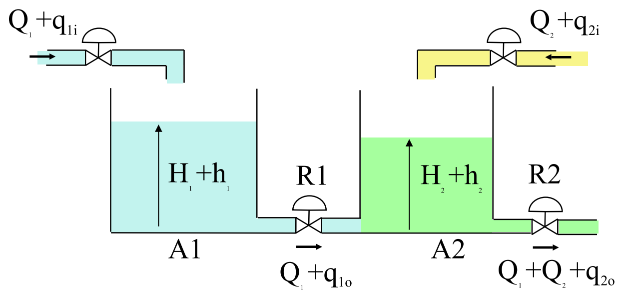


Figure 3: Two Tanks

Solved Problem 3.8. *Contributed by - James Welsh, University of Newcastle, Australia.*

Consider the two tanks system shown in Figure 3:

Q_1 & Q_2 *are steady state flows*
H_1 & H_2 *are steady state heights (head)*
R_1 & R_2 *are value resistances*
All lower case variables are considered to be small quantities.

Find a state space model for the system using h_1 and h_2 as the state variables and with q_{1i} and q_{2i} as the inputs.

Solutions to Solved Problem 3.8

Solved Problem 3.9. *Contributed by - Alvaro Liendo, Universidad Tecnica Federico Santa Maria, Chile.*

Build a linear model around the equilibrium point defined by $u_Q = \sqrt{6}$ for the system:

$$\frac{d^2y(t)}{dt^2} + y(t)\frac{dy(t)}{dt} + y^3(t) - y(t) = 2\frac{du(t)}{dt} + u^2(t) \tag{8}$$

Solutions to Solved Problem 3.9

Solved Problem 3.10. *Contributed by - Alvaro Liendo, Universidad Tecnica Federico Santa Maria, Chile.*

Build a state space model for the system with input $u(t)$ and output $y(t)$ and having a model given by the differential equation:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + y(t) = 2u(t) \tag{9}$$

Solutions to Solved Problem 3.10

Solved Problem 3.11. *Contributed by - Alvaro Liendo, Universidad Tecnica Federico Santa Maria, Chile.*

Build a state space model for the system with input $u(t)$ and output $y(t)$ and having a model given by the differential equation

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + y(t) = 2\frac{du(t)}{dt} \tag{10}$$

Chapter 3 - Solutions to Solved Problems

Solution 3.1.

3.1.1 *The operating point* (u_Q, y_Q) *must satisfy*

$$(1+0.2y_Q)y_Q = u(t) + 0.2u_Q^3 \Longrightarrow 0.2y_Q^2 + y_Q - 3.6 = 0 \tag{11}$$

This yields two operating points, P_1 *and* P_2 *given by* $(2, -7.4244)$ *and* $(2, 2.4244)$ *respectively.*

3.1.2 *To obtain the linearized models we can proceed in many ways. For instance, we can apply the method outlined in section §3.10. To do that we define the state as* $x(t) = y(t)$*. We thus have* $x_Q = y_Q$ *and*

$$\dot{x}(t) = f(x(t), u(t)) = -x(t) - 0.2x(t)^2 + u(t) + 0.2u(t)^3 \tag{12}$$

$$y(t) = g(x(t), u(t)) = x(t) \tag{13}$$

If we define $x(t) = x_Q + \Delta x(t)$*,* $u(t) = u_Q + \Delta u(t)$*, then*

$$\frac{d\Delta x(t)}{dt} = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}} \Delta x(t) + \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}} \Delta u(t) = (-1-0.4x_Q)\Delta x(t) + (1+0.6u_Q)\Delta u(t) \tag{14}$$

$$\Delta y(t) = \Delta x(t) \tag{15}$$

We can also express this in input-output form as

$$\frac{d\Delta y(t)}{dt} + (1+0.4y_Q)\Delta y(t) = (1+0.6u_Q)\Delta u(t) \tag{16}$$

For the two operating points described above, we have

$$P_1: \qquad \frac{d\Delta y(t)}{dt} - 1.9698\Delta y(t) = 2.2\Delta u(t) \tag{17}$$

$$P_2: \qquad \frac{d\Delta y(t)}{dt} + 1.9698\Delta y(t) = 2.2\Delta u(t) \tag{18}$$

Solution 3.2. *We first need to compute the state,* (x_{1Q}, x_{2Q})*, corresponding to the equilibrium point . We notice that* $x_{1Q} = y_Q = 0$*, and from the first state equation we have that*

$$0 = x_{2Q} + 3u_Q \Longleftarrow x_{2Q} = -3u_Q = 6 \tag{19}$$

The reader can readily verify that these values also satisfy the second state equation at the equilibrium point.

We next express the state and plant input, output as

$$x_1(t) = x_{1Q} + \Delta x_1(t); \quad x_2(t) = x_{2Q} + \Delta x_2(t) \tag{20}$$

$$u(t) = u_Q + \Delta u(t); \quad y(t) = y_Q + \Delta y(t) \tag{21}$$

and we finally use the method presented in section §3.10 of the book, leading to

$$\frac{d\Delta x_1(t)}{dt} = -2x_{1Q}\Delta x_1(t) + \Delta x_2(t) + 3\Delta u(t) = \Delta x_2(t) + 3\Delta u(t) \tag{22}$$

$$\frac{d\Delta x_2(t)}{dt} = -2x_{2Q}\Delta x_1(t) - 2x_{1Q}\Delta x_2(t) = -12\Delta x_1(t) \tag{23}$$

$$\Delta y(t) = \Delta x_1(t) \tag{24}$$

Solution 3.3. *The interesting aspect of this problem is the two unit delay on the input.*

The state variables must include all information we require to know at time $k = k_o$ such that, given the input $u[k]$, for all $k \geq k_o$, we are able to compute $y[k]$, for all $k > k_o$. From the system equation we have

$$y[k_o + 1] = u[k_o - 1] - 0.4y[k_o] \tag{25}$$

$$y[k_o + 2] = u[k_o] - 0.4y[k_o + 1] \tag{26}$$

We see that we require to know $y[k_o]$, and $u[k_o - 1]$ to predict the future response. We thus choose

$$x_1[k] = y[k]; \quad x_2[k] = u[k-1]; \tag{27}$$

and we notice that

$$x_1[k+1] = y[k+1] = u[k-1] - 0.4y[k] = x_2[k] - 0.4x_1[k] \tag{28}$$

$$x_2[k+1] = u[k] \tag{29}$$

$$y[k] = x_1[k] \tag{30}$$

Setting the above in matrix form, we finally obtain

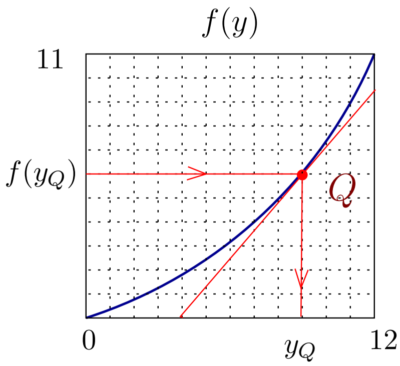
$$\begin{bmatrix} x_1[k+1] \\ x_2[k+1] \end{bmatrix} = \begin{bmatrix} -0.4 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[k] \tag{31}$$

$$y[k] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} \tag{32}$$

Solution 3.4. *The linearized model has the form*

$$\frac{d\Delta y(t)}{dt} + \left.\frac{df(y)}{dy}\right|_Q \Delta y(t) = 2\Delta u(t) \tag{33}$$

where $y(t) = y_Q + \Delta y(t)$ and $u(t) = u_Q + \Delta u(t)$ We thus need to compute the derivative of $f(y)$ at the equilibrium point. This is done graphically as shown below



We note that, at an equilibrium point, $f(y_Q) = 2u_Q$. Thus, in our particular example, $f(y_Q) = 6$. With this value we find the operating point. The slope of the tangent at that point is the derivative of $f(y)$ at Q, and is approximately equal to 1.19. Hence the linearized model is given by

$$\frac{d\Delta y(t)}{dt} + 1.19\Delta y(t) = 2\Delta u(t) \tag{34}$$

Solution 3.5.

3.5.1 *The number of states required for the system is equal to the number of initial conditions we can arbitrarily set in the network. In this case, the number is three: an initial current in inductor L_1 and the initial voltages on C_1 and C_2. The reader may note that the answer can be different if we allow some singular cases. For instance, if we make $R_4 = 0$, then the number of states required will be only two, since then the voltage in C_3 will always be equal to the source voltage and hence it can not be set arbitrarily.*

3.5.2 *To build the state space model we choose as state variables the electric signals $i_1(t)$, $v_2(t)$ and $v_3(t)$ which are shown in the network schematic shown in Figure 4.*

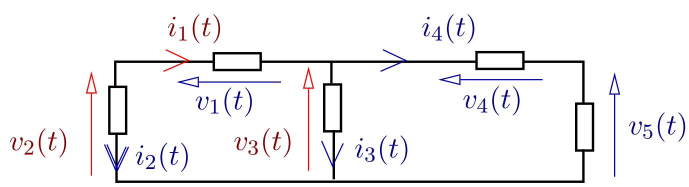


Figure 4: Electric network skeleton

Applying Kirchoff's laws and component laws we obtain

$$v_5(t) = v_{f5}(t) = -v_4(t) + v_3(t) = -R_4 i_4(t) + v_3(t) \tag{35}$$

$$i_4(t) = i_1(t) - i_3(t) = i_1(t) - C_3\frac{dv_3(t)}{dt} \tag{36}$$

$$v_2(t) = v_1(t) + v_3(t) = L_1\frac{di_1(t)}{dt} + v_3(t) \tag{37}$$

$$i_1(t) = -i_2(t) = -C_2\frac{dv_2(t)}{dt} \tag{38}$$

We also notice that the system input is the voltage of the independent voltage source, $v_{f5}(t)$. Rearranging the previous equations we finally obtain

$$\frac{di_1(t)}{dt} = \frac{1}{L_1}v_2(t) - \frac{1}{L_1}v_3(t) \tag{39}$$

$$\frac{dv_2(t)}{dt} = -\frac{1}{C_2}i_1(t) \tag{40}$$

$$\frac{dv_3(t)}{dt} = -\frac{1}{C_3}i_1(t) - \frac{1}{R_4C_3}v_3(t) + \frac{1}{R_4C_3}v_{f5}(t) \tag{41}$$

Solution 3.6.

3.6.1 *The volume of liquid in the tank is*

$$V = Ah \tag{42}$$

The rate of change of liquid in the tank is

$$\frac{dV}{dt} = u - f_{out} \tag{43}$$

$$A\frac{dh}{dt} = u - K\sqrt{h} \tag{44}$$

$$or \qquad \frac{dh}{dt} = -\frac{K}{A}\sqrt{h} - \frac{1}{A}u \tag{45}$$

3.6.2 *If $h = h^*$, then $u = K\sqrt{h^*}$ for a steady height.*
Let $\tilde{h} = h - h^*$, $\tilde{u} = u - u^*$.
Then linearizing (44) we obtain

$$A\frac{d(h^* + \tilde{h})}{dt} = u^* + \tilde{u} - K\sqrt{h^* + \tilde{h}} \tag{46}$$

$$\simeq u^* + \tilde{u} - K\sqrt{h^*} - \frac{K}{2}(h^*)^{\frac{1}{2}}\tilde{h} \tag{47}$$

Hence

$$\frac{d}{dt}\tilde{h} \simeq -\frac{K}{A2\sqrt{h^*}}\tilde{h} - \frac{1}{A}\tilde{u} \tag{48}$$

3.6.3 *Here*

$$V = Ah = ch^2 \tag{49}$$

Hence

$$\frac{dV}{dt} = u - f_{out} \tag{50}$$
$$c\frac{dh^2}{dt} = u - K\sqrt{h} \tag{51}$$
$$2ch\frac{dh}{dt} = u - K\sqrt{h} \tag{52}$$
$$\frac{dh}{dt} = -\frac{K}{2c\sqrt{h}} + \frac{u}{2ch} \tag{53}$$

Linearizing about h^* gives

$$u^* = K\sqrt{h^*} \qquad \textit{as before} \tag{54}$$

(Note that this is reasonable since we have to balance the outflow by the inflow).

Also

$$\frac{d\tilde{h}}{dt} \simeq -\frac{K}{2c\sqrt{h^*}} + \frac{K}{4c}(h^*)^{-\frac{3}{2}}\tilde{h} - \frac{u^*}{2ch^*} - \frac{\tilde{u}}{2ch^*} - \frac{u^*}{2c}(h^*)^{-2}\tilde{h} \tag{55}$$
$$= \left[\frac{K}{4c}(h^*)^{-\frac{3}{2}} - \frac{u^*}{2c}(h^*)^{-2}\right]\tilde{h} - \frac{\tilde{u}}{2ch^*} \tag{56}$$

Solution 3.7. *We assume that the cone makes an angle θ with the horizontal plane. Also, assume that the diameter of the base of the cone is d_o. Also let h denote the height of the ball.*

Then resolving the forces on the ball tangential to the wall, we have

$$mg\cos\theta = m\left\{cot[h + \frac{d_o}{2}\tan\theta]\right\}\omega^2\sin\theta \tag{57}$$

Differentiation with respect to time gives

$$-csc[[h + \frac{d_o}{2}\tan\theta]^2\omega^2\sin\theta\dot{h} + cot[h + \frac{d_o}{2}\tan\theta]2\omega\dot{\omega}\sin\theta = 0 \tag{58}$$

Solution 3.8. *Because we are only interested in small variations, we can assume that flow through the valves is linearly related to the difference in head.*

The equations for Tank 1 then become:

$$A_1\frac{dh_1}{dt} = q_{1i} - q_{1o} \tag{59}$$
$$q_{1o} = \frac{h_1 - h_2}{R_1} \tag{60}$$

The corresponding equations for Tank 2 are

$$A_2 \frac{dh_2}{dt} = q_{1o} + q_{2i} - q_{2o} \tag{61}$$

$$q_{2o} = \frac{h_2}{R_2} \tag{62}$$

Substituting 60 into 59

$$\frac{dh_1}{dt} = \frac{1}{A_1}\left(q_{1i} - \frac{h_1 - h_2}{R_1}\right) \tag{63}$$

Substituting 60 and 62 into 61

$$\frac{dh_2}{dt} = \frac{1}{A_2}\left(\frac{h_1 - h_2}{R_1} + q_{2i} - \frac{h_2}{R_2}\right) \tag{64}$$

We now define the state variables as

$$x_1 = h_1 \tag{65}$$

$$x_2 = h_2 \tag{66}$$

We then write 63 and 64 respectively as

$$\dot{x_1} = -\frac{1}{R_1 A_1} x_1 + \frac{1}{R_1 A_1} x_2 + \frac{1}{A_1} q_{1i} \tag{67}$$

$$\dot{x_2} = -\frac{1}{R_1 A_2} x_1 - \left(\frac{1}{R_1 A_2} + \frac{1}{R_2 A_2}\right) x_2 + \frac{1}{A_2} q_{2i} \tag{68}$$

In summary, the state space model is

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1 A_1} & \frac{1}{R_1 A_1} \\ \frac{1}{R_1 A_2} & -\left(\frac{1}{R_1 A_2} + \frac{1}{R_2 A_2}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{A_1} & 0 \\ 0 & \frac{1}{A_2} \end{bmatrix} \begin{bmatrix} q_{1i} \\ q_{2i} \end{bmatrix} \tag{69}$$

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{70}$$

Solution 3.9. *At equilibrium* $\frac{d^n y(t)}{dt^n}$ *and* $\frac{d^n y(t)}{dt^n}$ *are zero. Hence at the equilibrium point, the differential equation reduces to*

$$y_Q^3 - y_Q = u_Q^2 \tag{71}$$

This leads to $(u_Q, y_Q) = (\sqrt{6}, -2)$. *The other 2 solutions of the third degree equation are complex numbers.*

The linearized model is obtained using a first order Taylor series approximation , which is as follows:

$$f(x(t),y(t)) \approx f(x_Q,y_Q) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q\\ y=y_Q}} \Delta x + \left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_Q\\ y=y_Q}} \Delta y \tag{72}$$

Considering $u(t) = u_Q + \Delta u(t)$ *and* $y(t) = y_Q + \Delta y(t)$, *we obtain*

$$y(t)\frac{dy(t)}{dt} \approx y_Q\frac{d\Delta y(t)}{dt} \tag{73}$$
$$y^3(t) \approx 3y_Q^2\Delta y(t) + y_Q^3 \tag{74}$$
$$u^2(t) \approx 2u_Q\Delta u(t) + u_Q^2 \tag{75}$$

Thus

$$\frac{d^2\Delta y(t)}{dt^2} + y_Q\frac{d\Delta y(t)}{dt} + 3y_Q^2\Delta y(t) + y_Q^3 - \Delta y(t) - y_Q = 2\frac{d\Delta u(t)}{dt} + 2u_Q\Delta u(t) + u_Q^2 \tag{76}$$

Using equation (71) *and the value of* u_Q *and* y_Q *the linear model is*

$$\frac{d^2\Delta y(t)}{dt^2} - 2\frac{d\Delta y(t)}{dt} + 12\Delta y(t) - \Delta y(t) = 2\frac{d\Delta u(t)}{dt} + 2\sqrt{6}\Delta u(t) \tag{77}$$

Solution 3.10. *Consider the states* $x_1(t) = \frac{dy(t)}{dt}$ *and* $x_2(t) = y(t)$. *We then have the differential equation*

$$\dot{x}_1(t) + 3x_1(t) + x_2(t) = 2u(t) \tag{78}$$

Thus a suitable state space model is given by

$$\dot{x}_1(t) = -3x_1(t) - x_2(t) + 2u(t) \tag{79}$$
$$\dot{x}_2(t) = x_1(t) \tag{80}$$
$$y(t) = x_2(t) \tag{81}$$

Solution 3.11. *Consider the states* $\dot{x}_2(t) = x_1(t)$ *and* $x_1(t) = y(t)$ *With these states, the differential equation can be written*

$$\begin{aligned}\ddot{x}_1(t) + 3\dot{x}_1(t) + x_1(t) = 2\dot{u}(t) &\implies \ddot{x}_1(t) + 3\dot{x}_1(t) + \dot{x}_2(t) = 2\dot{u}(t)\\ &\implies \dot{x}_1(t) + 3x_1(t) + x_2(t) = 2u(t)\end{aligned} \tag{82}$$

Thus a suitable state space model is

$$\dot{x}_1(t) = -3x_1(t) - x_2(t) + 2u(t) \tag{83}$$
$$\dot{x}_2(t) = x_1(t) \tag{84}$$
$$y(t) = x_1(t) \tag{85}$$
