Chapter 4 - Solved Problems

Solved Problem 4.1. *Contributed by - James Welsh, University of Newcastle, Australia.*
Determine the transfer function of a linear time invariant system given the following information:

4.1.1 *The system has relative degree 3.*

4.1.2 *It has 3 poles of which 2 are at -2 and -4.*

4.1.3 *The impulse response resembles a step response for a stable linear system with a steady state value of 0.25.*

Solutions to Solved Problem 4.1

Solved Problem 4.2. *Consider a system with transfer function given by*

$$H(s) = \frac{2(s+1)}{s^2+4s+9} \tag{1}$$

and denote by $g(t)$ the unit step response with zero initial conditions.
Compute the initial slope of $g(t)$, i.e. the slope at $t = 0_+$.

Solutions to Solved Problem 4.2

Solved Problem 4.3. *The unit step response of a system with zero initial conditions is given by*

$$y(t) = 5 - 2e^{-3t} - 3e^{-6t} \tag{2}$$

Compute the system zeros, if any.

Solutions to Solved Problem 4.3

Solved Problem 4.4. *The response of a linear system to a Dirac delta is given by $h(t) = 2e^{-3t}\mu(t)$. Compute the system d.c. gain*

Solutions to Solved Problem 4.4

Solved Problem 4.5. *A signal $f(t)$ has the Fourier transform given by $F(j\omega)$, depicted in Figure 1.*

4.5.1 *Show that $f(t)$ is an even real signal.*

4.5.2 *Compute $f(0)$, without inverting the Fourier transform.*

Solutions to Solved Problem 4.5

Solved Problem 4.6. *A system has a transfer function given by $H(s) = \dfrac{2}{s+1} + \dfrac{\boldsymbol{\alpha}}{s+2}$. Is there a range of real values for $\boldsymbol{\alpha}$ such that the system unit step response exhibits undershoot?*

Solutions to Solved Problem 4.6

Solved Problem 4.7. *Contributed by - Rodrigo Musalem, Universidad Tecnica Federico Santa Maria, Chile.*
Determine the Laplace Transform for each of the following functions.

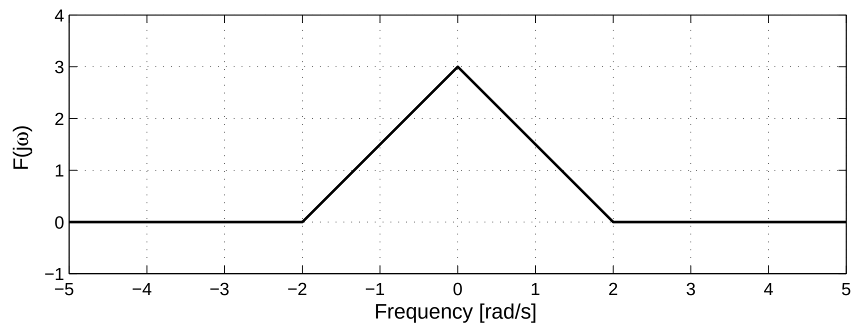

Figure 1: Spectrum of an even signal

4.7.1

$$f_1(t) = -\frac{1}{2}te^{-t} + \frac{5}{4}e^{-t} - \frac{5}{4}e^{-3t} \tag{3}$$

4.7.2

$$f_2(t) = \frac{1}{10}t\sin 5t \tag{4}$$

4.7.3

$$f_3(t) = 8t\cos 3t + \frac{1}{9}\left(1 - e^{-2t} - 2te^{-2t}\right) \tag{5}$$

4.7.4

$$f_{4i}(t) = e^{-3t}\mu(t-2) \tag{6}$$

$$f_{4ii}(t) = e^{-3(t-2)}\mu(t-2) \tag{7}$$

Compare and discuss. Then, analyze the general case,

$$f_1(t) = f(t)\mu(t-\tau) \tag{8}$$
$$f_2(t) = f(t-\tau)\mu(t-\tau) \tag{9}$$

Solutions to Solved Problem 4.7

Solved Problem 4.8. *Contributed by - Rodrigo Musalem, Universidad Tecnica Federico Santa Maria, Chile.*

Determine the Inverse Laplace Transform for each of the following functions.

4.8.1

$$F_1(s) = \frac{3(s+2)}{s(s+3)(s+4)} \tag{10}$$

4.8.2

$$F_2(s) = \frac{16}{(s^2+4)^2} \tag{11}$$

4.8.3

$$F_3(s) = e^{-2s} F_2(s) \tag{12}$$

4.8.4

$$F_4(s) = \frac{s + \alpha + \varepsilon}{(s + \alpha)(s + 1)} \tag{13}$$

Where $|\varepsilon| \ll |\alpha|$. *Analyze your results.*

4.8.5

$$F_5(s) = \frac{s + a}{s^2} \tag{14}$$

Solutions to Solved Problem 4.8

Solved Problem 4.9. *Contributed by - Rodrigo Musalem, Universidad Tecnica Federico Santa Maria, Chile.*

Obtain the transfer function $\frac{Y(s)}{U(s)}$ *for the differential equation below.*

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y(t) = 2\frac{du}{dt} + u(t) \tag{15}$$

Solutions to Solved Problem 4.9

Solved Problem 4.10. *Contributed by - Rodrigo Musalem, Universidad Tecnica Federico Santa Maria, Chile.*

Obtain the Inverse Laplace Transform for the following function.

$$F(s) = \frac{As + B}{(s + \alpha)^2 + \beta^2} \tag{16}$$

Solutions to Solved Problem 4.10

Chapter 4 - Solutions to Solved Problems

Solution 4.1.

4.1.1 *The system has relative degree 3 with 3 poles - hence it has no finite zeros.*

4.1.2 *It has 3 poles, hence it takes the form*

$$G(s) = \frac{K}{A'(s)(s+2)(s+4)} \tag{17}$$

4.1.3 *Since the impulse response resembles a step response with steady state value we conclude the system must contain a pole at zero.*

Therefore the transfer function is of the form

$$G(s) = \frac{K}{s(s+2)(s+4)} \tag{18}$$

We use the final value theorem to determine K.

$$\lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{sK}{s(s+2)(s+4)} \tag{19}$$

$$= \frac{K}{8} \tag{20}$$

Given the steady state value of 0.25, i.e., $\lim_{s\to 0} sG(s) = 0.25$ *we find*

$$K = 2 \tag{21}$$

Therefore

$$G(s) = \frac{2}{s(s+2)(s+4)} \tag{22}$$

Solution 4.2. *We first note that the derivative of $g(t)$ is the response to the derivative of the unit step, i.e., it is equal to the response to a Dirac delta. Hence*

$$\mathcal{L}\left[\frac{dg(t)}{dt}\right] = \mathcal{L}\left[h(t)\right] = H(s) \tag{23}$$

Therefore the answer to the question translates into computing $h(0_+)$. This can be computed via the Initial Value Theorem (see Table 4.2 of the book). This yields

$$\dot{g}(0_+) = h(0_+) = \lim_{s\to\infty} sH(s) = \lim_{s\to\infty} \frac{2s(s+1)}{s^2+4s+9} = 2 \tag{24}$$

Solution 4.3. *We first notice that*

$$\mathcal{L}\left[y(t)\right] = Y(s) = H(s)\frac{1}{s} \Longrightarrow H(s) = sY(s) \tag{25}$$

Hence

$$H(s) = s\left[\frac{5}{s} - \frac{2}{s+3} - \frac{3}{s+6}\right] = 6\frac{4s+15}{s(s+3)(s+6)} \tag{26}$$

Thus the system has one zero located at $s = -\frac{15}{4}$

Solution 4.4. *The system d.c. gain is given by* $H(0)$ *where* $H(s)$ *is the system transfer function, which is equal the Laplace transformation of* $h(t)$*, i.e. the Laplace transform of the system response to a unit impulse. We also note that, for this system, the Laplace transform of* $h(t)$ *has a region of convergence given by* $\Re\{s\} > -3$*. Therefore*

$$H(0) = \int_0^\infty h(t)\,dt = \int_0^\infty 2e^{-3t}\,dt = \frac{2}{3} \tag{27}$$

Solution 4.5. *We first note that Figure 1 depicts* $F(j\omega)$*. This implies that the angle of the Fourier transform is 0 (or a multiple of* 2π*), for all* $\omega \in \mathbb{R}$*.*

4.5.1 *We recall from subsection §4.10.1 of the book that*

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\,d\omega \tag{28}$$

Hence, if we conjugate both sides of equation (28) and use the fact that $F(j\omega)^* = F(-j\omega)$ *we obtain*

$$f(t)^* = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(-j\omega)e^{-j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\,d\omega = f(t) \tag{29}$$

The above result says that $f(t)$ *is real since* $f(t)^* = f(t)$ *for all* t*.*

From (28) we also have that

$$f(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{-j\omega t}\,d\omega \tag{30}$$

We now make a change of variables in the integral, say $\omega = -v$*, then*

$$f(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(-jv)e^{jvt}\,d\omega \tag{31}$$

However, since $F(j\omega)$ *is real and even, then* $F(-jv) = F(jv)$*. This proves that* $f(t) = f(-t)$*, i.e., that* $f(t)$ *is an even real signal.*

4.5.2 *To compute* $f(0)$ *we use (28) which yields*

$$f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\,d\omega = \frac{1}{\pi}\int_0^2 F(j\omega)\,d\omega = \frac{3}{\pi} \tag{32}$$

Solution 4.6. *We first note that if* $\boldsymbol{\alpha} > 0$ *the unit step response is always positive, thus no undershoot occurs.*

To have undershoot in the unit step response it is sufficient that a linear system have a real zero in the open RHP (a NMP zero, see page 99 of the book). To investigate that possibility we add the two fractions in $H(s)$ *to yield*

$$H(s) = \frac{2s + 4 + \boldsymbol{\alpha} s + \boldsymbol{\alpha}}{(s+1)(s+2)} = (4 + \boldsymbol{\alpha}) \frac{\dfrac{2+\boldsymbol{\alpha}}{4+\boldsymbol{\alpha}} s + 1}{(s+1)(s+2)} \tag{33}$$

Then we have a NMP zero if and only if $-4 < \boldsymbol{\alpha} < -2$.

Solution 4.7. In each case, we first obtain the solution by hand and we then show the MAPLE code to obtain the same result.

4.7.1 Applying the Laplace transform to each term in $f_1(t)$ we obtain,

$$F_1(s) = -\frac{\frac{1}{2}}{(s+1)^2} + \frac{\frac{5}{4}}{s+1} - \frac{\frac{5}{4}}{s+3} \tag{34}$$

After some algebraic work, we can write $F_1(s)$ as,

$$F_1(s) = \frac{2s+1}{(s+1)^2(s+3)} \tag{35}$$

The corresponding MAPLE code to solve this problem is,

```
>with(inttrans):
>simplify(laplace(-0.5*t*exp(-t)+(5/4)*exp(-t)-(5/4)*exp(-3*t),t,s));
```

This code also yields the result (35).

4.7.2 From Table 4.1 (in the book), we have the following Laplace Transform pair

$$\mathcal{L}\left[\frac{1}{2\omega} t \sin \omega t\right] = \frac{s}{(s^2+\omega^2)^2} \tag{36}$$

Applying this Transform, we obtain

$$F_2(s) = \frac{s}{(s^2+25)^2} \tag{37}$$

The above result can also be obtained using MAPLE using,

```
>with(inttrans):
>laplace((1/10)*t*sin(5*t), t, s);
```

4.7.3 Similarly to the previous problem, we can use the Laplace Transform pairs

$$\mathcal{L}\left[t\cos\omega t\right] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2} \tag{38}$$

$$\mathcal{L}\left[\frac{1}{a^2}(1-e^{-at}-ate^{-at})\right] = \frac{1}{s(s+a)^2} \tag{39}$$

to obtain,

$$F_3(s) = 8\frac{s^2-9}{(s^2+9)^2} + \frac{\frac{4}{9}}{s(s+2)^2} \tag{40}$$

This result can also be derived with MAPLE, using

```
>with(inttrans):
>laplace(8*t*cos(3*t)+(1/9)*(1-exp(-2*t)-2*t*exp(-2*t)), t, s);
```

4.7.4 We can use the following Laplace Transform property

$$\mathcal{L}\left[f(t-\tau)\mu(t-\tau)\right] = e^{-s\tau}F(s) \qquad \forall\tau>0 \tag{41}$$

We also note that we can write $f_{4i}(t)$ as,

$$f_{4i}(t) = \frac{e^6}{e^6}e^{-3t}\mu(t-2) = \frac{1}{e^6}e^{-3(t-2)}\mu(t-2) \tag{42}$$

Therefore, applying the property (41) we obtain,

$$F_{4i}(s) = \frac{1}{e^6}e^{-2s}\frac{1}{s+3} \tag{43}$$

On the other hand, for the case of $f_{4ii}(t)$ is possible to use directly property (41), to obtain

$$F_{4ii}(s) = e^{-2s}\frac{1}{s+3} \tag{44}$$

These results can also be obtained using MAPLE, via

```
>with(inttrans):
>f_4i:=laplace(exp(-3*t)*Heaviside(t-2), t, s);
>f_4ii:=laplace(exp(-3*(t-2))*Heaviside(t-2), t, s);
>simplify(f_4ii);
```

Note that in this case it was possible to write the function f_{4i} so we could use the property described by (41). As a consequence, the Laplace Transform of f_{4i} and f_{4ii} only differ by a factor $\frac{1}{e^6}$. However, in general it is not possible to build a direct relationship between Laplace Transforms of $f(t)\mu(t-\tau)$ and $f(t-\tau)\mu(t-\tau)$.

To better appreciate the situation, consider the function $f(t)$ shown in Figure 2 and $f(t)\mu(t-\tau)$ as shown in Figure 3, where $\tau = 2$.

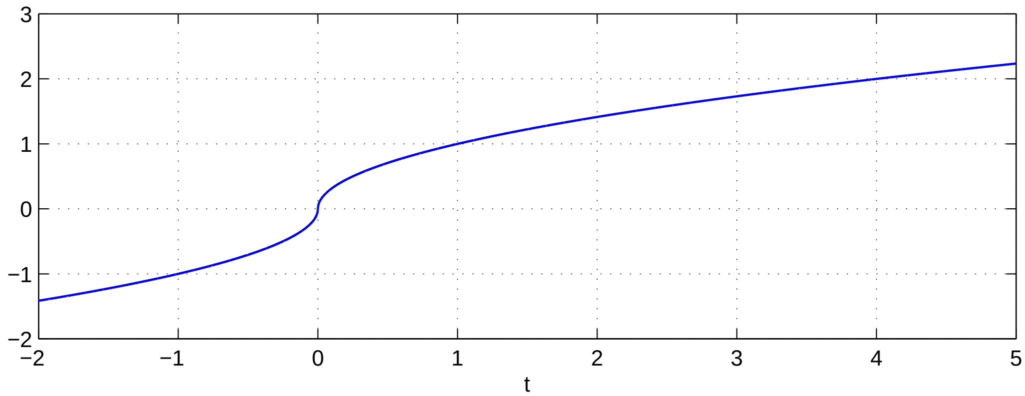

Figure 2: Arbitrary function $f(t)$

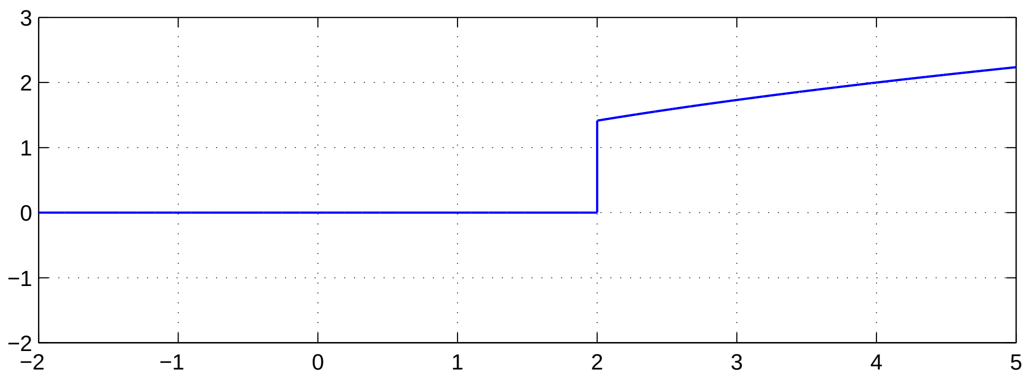

Figure 3: Graph of $f(t)\mu(t-\tau)$, with $\tau = 2$

The function $f(t-\tau)\mu(t-\tau)$ is shown in Figure 4. Note that with $\tau = 2$,this new function has information which can not be obtained from Figure 3. This illustrates why, in general, it is not possible to build a direct relationship between the two cases.

Solution 4.8. *In these problems we first use the Laplace Transform tables, and we then present the MAPLE code which yields the same solution.*

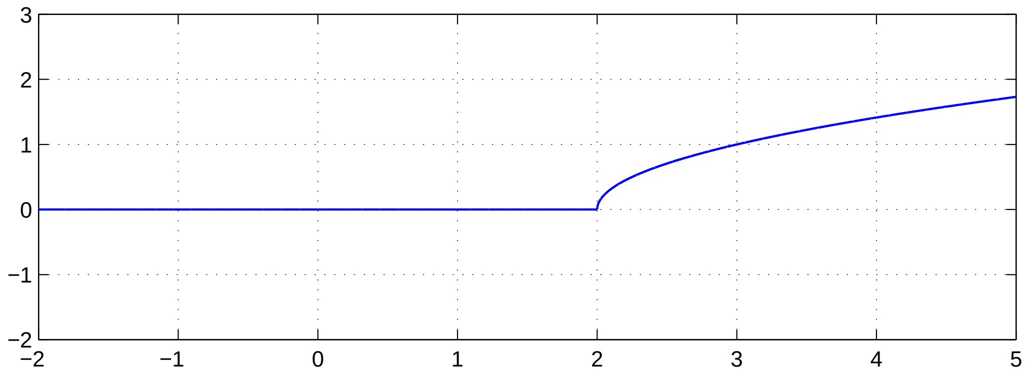

Figure 4: Graph of $f(t-\tau)\mu(t-\tau)$, with $\tau = 2$

4.8.1 *We first expand the function using partial fractions,*

$$\frac{3(s+2)}{s(s+3)(s+4)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+4} \tag{45}$$

$$3(s+2) = A(s+3)(s+4) + Bs(s+4) + Cs(s+3) \tag{46}$$

We can evaluate this expression for certain values of s to find the constants A, B and C.

Taking $s = -3$, $s = -4$ and $s = 0$ successively we find that $A = \frac{1}{2}$, $B = 1$ and $C = -\frac{3}{2}$.

Thus, applying the Inverse Laplace Transform to (45) with the values already computed for A, B and C, we have,

$$f_1(t) = \left(\frac{1}{2} + e^{-3t} - \frac{3}{2}e^{-4t}\right)\mu(t) \tag{47}$$

The corresponding MAPLE code is,

```
>with(inttrans):
>invlaplace((3*(s+2))/(s*(s+3)*(s+4)),s,t);
```

4.8.2 *To answer this question, we can use the Laplace Transform pair*

$$\mathcal{L}^{-1}\left[\frac{2\omega^3}{(s^2+\omega^2)^2}\right] = \sin\omega t - \omega t\cos\omega t \tag{48}$$

We can then apply this property directly, using $\omega = 2$, and obtain

$$f_2(t) = (\sin 2t - 2t\cos 2t)\,\mu(t) \tag{49}$$

In MAPLE, the corresponding code is

```
>with(inttrans):
>invlaplace(16/(s^2+4)^2, s, t);
```

4.8.3 *The (time) function is a delayed version of the previous case(see Table 4.1 in the book). Therefore, its Inverse Laplace Transform is*

$$f_3(t) = (\sin 2(t-2) - 2(t-2)\cos 2(t-2))\,\mu(t-2) \tag{50}$$

In MAPLE, the code is

```
>with(inttrans):
>invlaplace((exp(-2*s))*(16/(s^2+4)^2), s, t);
```

4.8.4 *Expanding the function $F_4(s)$ using partial fractions we have*

$$\frac{s+\alpha+\varepsilon}{(s+\alpha)(s+1)} = \frac{A}{s+\alpha} + \frac{B}{s+1} \tag{51}$$

where, $A = \dfrac{\varepsilon}{1-\alpha}$ *and* $B = \dfrac{\alpha+\varepsilon-1}{\alpha-1}$.

Applying the Inverse Laplace Transform of each of these fractions, we have

$$f_4(t) = \left(\frac{\varepsilon}{1-\alpha}e^{-\alpha t} + \frac{\alpha-1+\varepsilon}{\alpha-1}e^{-t}\right)\mu(t) \tag{52}$$

In MAPLE,

```
>with(inttrans):
>invlaplace((s+a+e)/((s+a)*(s+1)),s,t);
```

Since $|\varepsilon| \ll |\alpha|$, the step response for this function will contain the mode $e^{-\alpha t}$ with a small weighting. This is due to a near cancellation between the pole located at $-\alpha$ and the zero located at $-(\alpha+\varepsilon)$.

4.8.5 *Expanding the function using partial fractions we have,*

$$\frac{s+\alpha}{s^2} = \frac{A}{s} + \frac{B}{s^2} \tag{53}$$

where $A = 1$ *and* $B = \alpha$. *Applying the Inverse Laplace Transform we obtain,*

$$f_5(t) = (1 + \alpha t)\,\mu(t) \tag{54}$$

The corresponding code in MAPLE is,

```
>with(inttrans):
>invlaplace((s+a)/s^2,s,t);
```

Solution 4.9. *We take the Laplace transform of the differential equation. This leads to*

$$s^2Y(s) - sy(0^-) - \dot{y}(0^-) + 4sY(s) - 4y(0^-) + 3Y(s) = 2sU(s) - 2u(0^-) + U(s) \tag{55}$$

$$Y(s)\left(s^2 + 4s + 3\right) = U(s)(2s+1) - 2u(0^-) + sy(0^-) + \dot{y}(0^-) + 4y(0^-) \tag{56}$$

Setting the initial conditions to zero, shows that the transfer function is

$$\frac{Y(s)}{U(s)} = \frac{2s+1}{s^2+4s+3} \tag{57}$$

Solution 4.10. *We can combine the Laplace transforms for the sine and the cosine functions together with the frequency shift property (see Table 4.2 in the book) to obtain*

$$\mathcal{L}\left[e^{-at}\sin\omega t\right] = \frac{\omega}{(s+a)^2+\omega^2} \tag{58}$$

$$\mathcal{L}\left[e^{-at}\cos\omega t\right] = \frac{s+a}{(s+a)^2+\omega^2} \tag{59}$$

Next, it is possible to re-write $F(s)$ *as follows:*

$$\frac{As+B}{(s+\alpha)^2+\beta^2} = A\frac{\left(s+\frac{B}{A}+\alpha-\alpha\right)}{(s+\alpha)^2+\beta^2} = A\frac{s+\alpha}{(s+\alpha)^2+\beta^2} + \frac{B-\alpha A}{(s+\alpha)^2+\beta^2} \tag{60}$$

Multiplying and dividing the last term by β *we obtain,*

$$F(s) = A\frac{s+\alpha}{(s+\alpha)^2+\beta^2} + (B-\alpha A)\frac{1}{\beta}\frac{\beta}{(s+\alpha)^2+\beta^2} \tag{61}$$

Then, using known Laplace Transform pairs, we finally obtain

$$f(t) = \left(Ae^{-\alpha t}\cos\beta t + (B-\alpha A)\frac{1}{\beta}e^{-\alpha t}\sin\beta t\right)\mu(t) \tag{62}$$

The result can also be obtained with MAPLE, using the code

```
>with(inttrans):
>invlaplace((A*s+B)/((s+alfa)^2+beta^2),s,t);
```
