

Chapter 6 - Solved Problems

Solved Problem 6.1. *Contributed by - James Welsh, University of Newcastle, Australia.*

Find suitable values for the PID parameters using the Z-N tuning strategy for the nominal plant G_o , where

$$G_o(s) = \frac{e^{-s}}{s+1} \quad (1)$$

Solutions to Solved Problem 6.1

Solved Problem 6.2. *A plant has a nominal given by*

$$G_o(s) = \frac{1}{(s-1)^2} \quad (2)$$

Prove that this system cannot be stabilized with a PI controller

Solutions to Solved Problem 6.2

Solved Problem 6.3. *Show, using Root Locus analysis that the plant in Problem 6.2 can be stabilized using a PID controller.*

Solutions to Solved Problem 6.3

Solved Problem 6.4. *Consider a plant with nominal model given by*

$$G_o(s) = \frac{1}{s+2} \quad (3)$$

Compute the parameters of a PI controller so that the natural modes of the closed loop response decay at least as fast as e^{-5t} .

Solutions to Solved Problem 6.4

Solved Problem 6.5. *Assume that the Ziegler-Nichols ultimate gain method is used to tune a PID controller for a plant with model*

$$G_o(s) = 2 \frac{e^{-s}}{(2s+1)^2} \quad (4)$$

Determine the parameters of the PID controller.

Solutions to Solved Problem 6.5

Solved Problem 6.6. *Assume that the theory presented in section §6.5 of the book is used to tune a PI controller for a process with transfer function*

$$G(s) = \frac{24(s+1.5)e^{-0.2s}}{(s+2)^2(s+6)} \quad (5)$$

- 6.6.1** *Find the approximate model which underlies the tuning strategy. Compare its step response with that of the true process.*
- 6.6.2** *Compute the parameters of a PI controller as per Table 6.2 in the book.*
- 6.6.3** *Compare the performance of the PI controller acting on the model with that of PI controller acting on the true process.*

Solutions to Solved Problem 6.6

Chapter 6 - Solutions to Solved Problems

Solution 6.1. We first determine the critical frequency.

We set $s = j\omega$.

$$G_o(j\omega) = \frac{e^{-j\omega}}{j\omega + 1} \quad (6)$$

$$= \frac{\cos \omega - j \sin \omega}{j\omega + 1} \quad (7)$$

$$= \frac{(\cos \omega - j \sin \omega)(-j\omega + 1)}{\omega^2 + 1} \quad (8)$$

$$= \frac{1}{\omega^2 + 1}(-j\omega \cos \omega - \omega \sin \omega + \cos \omega - j \sin \omega) \quad (9)$$

$$= \frac{1}{\omega^2 + 1}(\cos \omega - \omega \sin \omega - j(\omega \cos \omega + \sin \omega)) \quad (10)$$

at the critical frequency, the imaginary term = 0.

$$\omega \cos \omega + \sin \omega = 0 \quad (11)$$

$$\omega = -\tan \omega \quad (12)$$

which when solved, for the smallest +ve value of ω , yields

$$\omega_c = 2.0288 \text{ rad/sec} \quad (13)$$

hence $P_c = 3.097$ seconds.

Now the critical gain occurs when

$$K_c G_o(j\omega_c) = -1 \quad (14)$$

$$\therefore K_c = \frac{-1}{G_o(j\omega_c)} \quad (15)$$

$$= 2.2619 \quad (16)$$

Using Table 6.1 in the book we find the PID parameters to be

$$K_p = 1.3571 \quad (17)$$

$$T_r = 1.548 \quad (18)$$

$$T_d = 0.3871 \quad (19)$$

We then choose

$$\tau_D = 0.1T_d \quad (20)$$

$$= 0.03871 \quad (21)$$

Solution 6.2. We consider the general form of a PI controller as

$$C(s) = \frac{as + b}{s} \quad (22)$$

Thus, the closed loop characteristic polynomial, $A_{cl}(s)$, is given by

$$A_{cl}(s) = \text{numerator of } \{1 + G_o(s)C(s)\} = s^3 - 2s^2 + (1 + a)s + b \quad (23)$$

Hence, there are no values for a and b such that $A_{cl}(s)$ is strictly Hurwitz. This originates from the fact that one of the coefficients of the polynomial $A_{cl}(s)$ is negative no matter what values we choose for a and b .

Solution 6.3. Since Root Locus analysis is to be used, the PID controller transfer function is first expressed in pole-zero form, i.e.,

$$C(s) = K \frac{(s - c_1)(s - c_2)}{s(s - p_2)} \quad (24)$$

Then, the open loop transfer function $G_o(s)C(s)$ has relative degree equal to 2, with zeros at c_1 and c_2 and poles at $p_1 = 0$, p_2 , $p_3 = 1$ and $p_4 = 1$. Thus implies that the Root Locus has two asymptotes with slopes $\pm \frac{\pi}{2}$ (for $K > 0$) and the asymptotes intersect at $(\sigma, 0)$, where

$$\sigma = \frac{\sum_{i=1}^4 p_i - \sum_{i=1}^2 c_i}{2} = \frac{2 + p_2 - c_1 - c_2}{2} \quad (25)$$

To ensure stability for a sufficiently large value of K , we choose the controller zeros and the controller pole so that $\sigma < 0$, this requires that

$$c_1 + c_2 > 2 + p_2 \quad (26)$$

Say we choose $p_2 = -12$, then (26) is satisfied if we choose, for instance $c_1 = -3$ and $c_2 = -4$. The Root Locus for $K > 0$ is shown in Figure 1.

The above results yield a PID controller transfer function

$$C(s) = K \frac{(s + 3)(s + 4)}{s(s + 12)} \quad (27)$$

Using the MATLAB command **rlocfind** we can compute the critical value, K_c , such that the closed loop is stable for all $K > K_c$. This is done using the MATLAB code

```
>> Go=tf(1,[1 -2 1]);Co=tf([1 7 12],[1 12 0]);GoCo=Go*Co;
>> rlocus(GoCo);
>> rlocfind(GoCo);
```

The last command generates a cross-hair that we move until it coincides with the imaginary axis crossing of the root locus. A mouse click delivers $K_c \approx 135$

Alternatively one can enter the MATLAB environment **rltool** with the command

```
>> rltool(Go,Co);
```

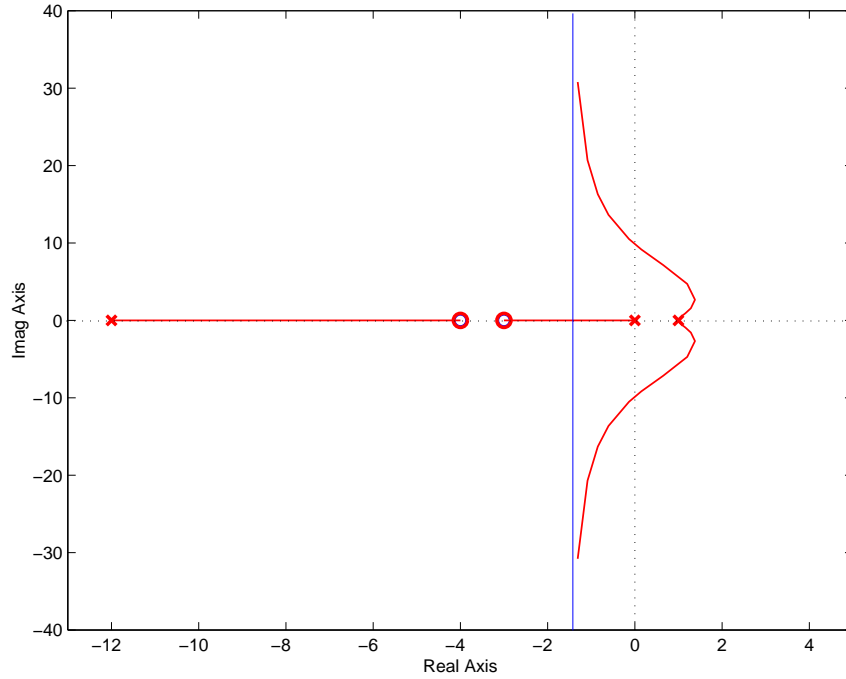


Figure 1: Root locus

Solution 6.4. A PI controller has transfer function given by

$$C(s) = \frac{as + b}{s}; \quad \text{where } a = K_p; \quad b = \frac{K_p}{T_r} \quad (28)$$

The the closed loop characteristic polynomial, $A_{cl}(s)$, is given by

$$A_{cl}(s) = \text{numerator of } \{1 + G_o(s)C(s)\} = s^2 + (2 + a)s + b \quad (29)$$

We choose the controller to obtain a pair of complex conjugate poles. To achieve a closed loop transient as fast as e^{-5t} , those poles must have real parts equal to -5 . This requires $a = 8$ and $b = 49$. Hence

$$C(s) = \frac{8s + 49}{s}; \quad (30)$$

The closed loop response to a unit step reference is shown in Figure 2.

Solution 6.5. We have to find the critical frequency, ω_c and the critical gain, K_c . This can be done using, for instance, the Bode diagrams shown in Figure 3.

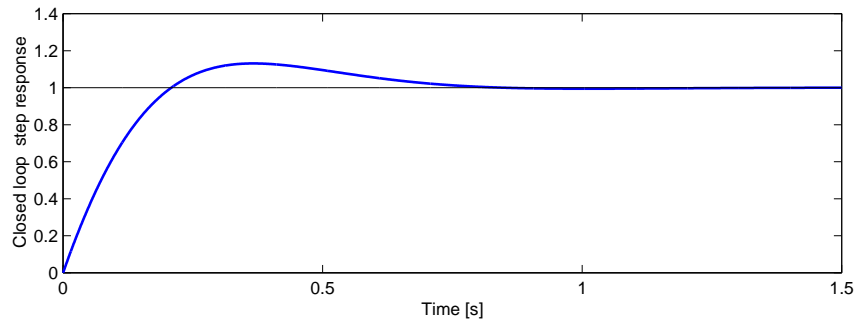


Figure 2: Closed loop unit step response

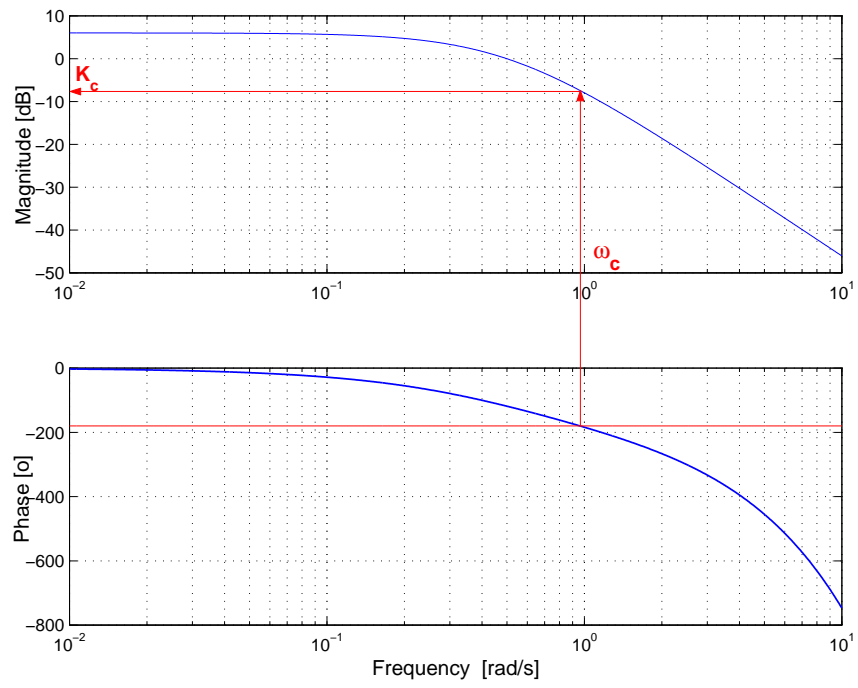


Figure 3: Bode diagrams

From the diagrams in Figure 3 we obtain $\omega_c = 0.96$ [rad/s] and $K_c = -6.5$ [dB] = 0.4732. The oscillation period is then $P_c = 2\pi/\omega_c = 6.55$. Finally, from Table 6.1 in the book, the parameters of the PID controller are

$$K_p = 0.6K_c = 0.2839; \quad T_r = 0.5P_c = 3.27; \quad T_d = 0.125P_c = 0.819 \quad (31)$$

Solution 6.6.

6.6.1 The methods based on using process step response implicitly use a (nominal) model having the general form

$$G_o(s) = \frac{K_o e^{-\tau_o s}}{\nu_o s + 1} \quad (32)$$

The model parameters K_o , τ_o and ν_o are obtained from the process step response. For the system in equation (5), the step response is as shown in Figure 4.

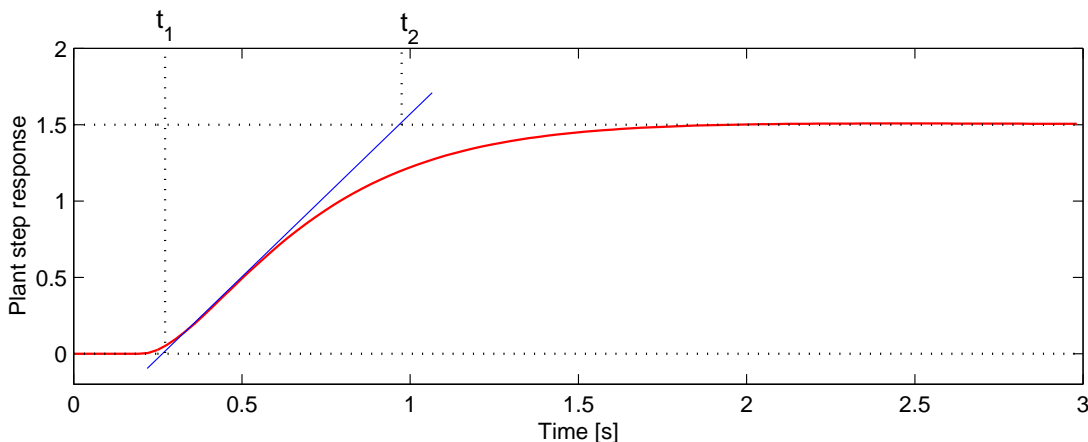


Figure 4: Plant response to an input $u(t) = \mu(t)$

From Figure 4 we have that $K_o = 1.5$, $\nu_o = t_2 - t_1 = 0.703$ and $\tau_o = t_1 = 0.272$ (use the MATLAB command **ginput**). Therefore the approximate model is

$$G_o(s) = \frac{1.5e^{-0.272s}}{0.703s + 1} \quad (33)$$

We next apply a unit step input to both, $G(s)$ and $G_o(s)$. The results are shown in Figure 5.

6.6.2 The controller parameters are then computed using Table 6.2 in the book employing the values obtained above for ν_o , K_o and $\tau_o = t_1$. This yields a controller given by

$$C(s) = K_p \left(1 + \frac{1}{T_r s} \right) = 1.55 \left(1 + \frac{1}{0.81s} \right) \quad (34)$$

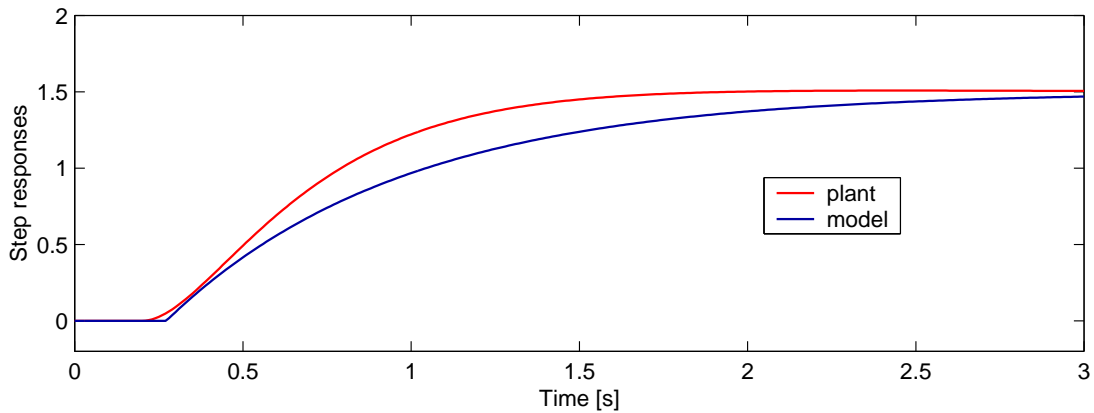


Figure 5: Comparison of the step response of the process and its model

6.6.3 We next compare the performance of the two control loops: one, where the controller is used to control the original process, and the other, where the controller is used to control a system having the nominal model (33). This is done using the SIMULINK schematic shown in Figure 6¹. In the upper loop the original process is considered, while in the lower loop we use the model (33). Both loops are driven by the same reference. For the simulation we use a unit step reference at $t = 1$ [s].

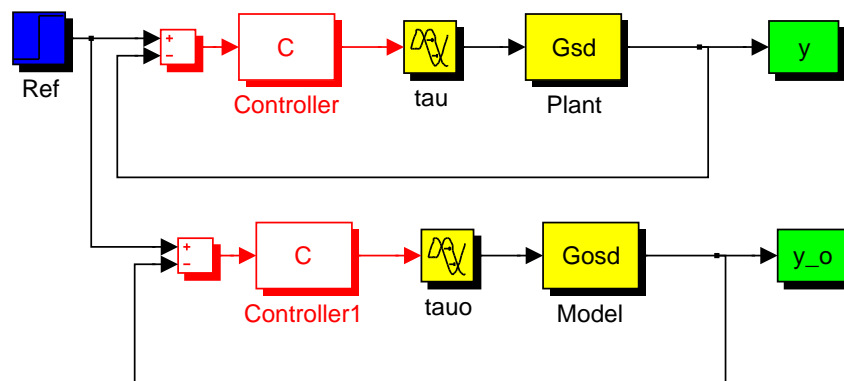


Figure 6: SIMULINK schematic

The simulation is run and the results are shown in Figure 7.

Figure 7 shows that better performance is achieved when the controller is acting on the nominal model (33). This can be explained by the fact that the approximate model turns out to have a larger rise time than the real process, as shown in Figure 5.

¹ The transfer function blocks C, Gsd and Gosd correspond to LTI Simulink blocks (Control System toolbox).

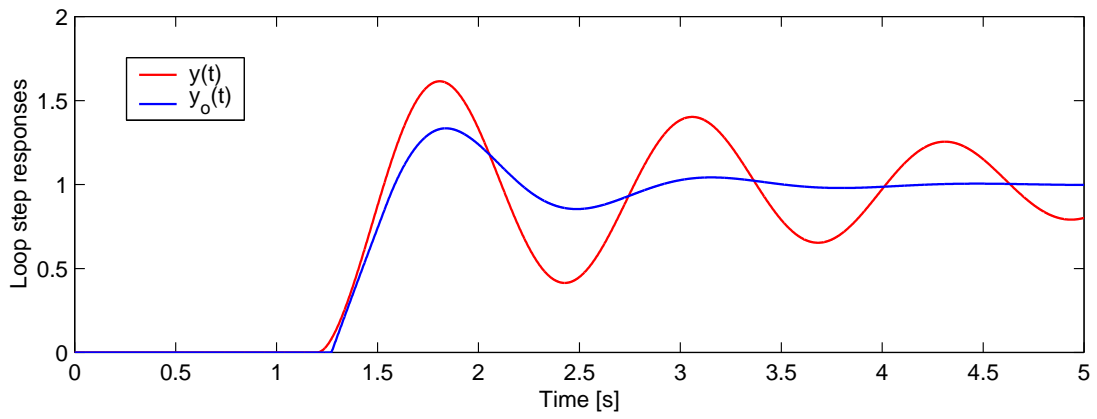


Figure 7: Closed loop step responses