Chapter 7 - Solved Problems

Solved Problem 7.1. *A continuous time system has transfer function $G_o(s)$ given by*

$$G_o(s) = \frac{B_o(s)}{A_o(s)} = \frac{2}{(s-1)(s+2)} = \frac{2}{s^2+s-2} \tag{1}$$

Find a controller of minimal complexity which stabilizes the plant, yields zero steady state error for step disturbances and generates a closed loop with natural modes which decay at least as fast as e^{-3t}.

Solutions to Solved Problem 7.1

Solved Problem 7.2. *Consider a plant with nominal model*

$$G_o(s) = \frac{B_o(s)}{A_o(s)} = \frac{2(s-1)}{(s-4)(s+2)} = \frac{2s-2}{s^2-2s-8} \tag{2}$$

7.2.1 *Using pole assignment synthesis build a controller with integration and with $A_{cl}(s)$ dominated[1] by a factor $(s+1)^2$. Plot the response to a unit step reference.*

7.2.2 *Repeat with $A_{cl}(s)$ dominated by a factor $(s+8)^2$.Plot the response to a unit step reference.*

7.2.3 *Discuss your results.*

Solutions to Solved Problem 7.2

Solved Problem 7.3. *A plant with nominal model*

$$G_o(s) = \frac{1}{(s+1)^2} \tag{3}$$

is in a feedback loop under control with a PI controller having transfer function

$$C(s) = \frac{s+0.7}{s} \tag{4}$$

Is is possible that this controller has resulted from a pole assignment synthesis.

Solutions to Solved Problem 7.3

Solved Problem 7.4. *A plant has nominal model*

$$G_o(s) = \frac{B_o(s)}{A_o(s)} = \frac{4}{(s+4)(s+2)} = \frac{4}{s^2+6s+8} \tag{5}$$

The reference is a constant. However there is a disturbance at the plant input. It has the general form

$$d_i(t) = K_1 \cos(\sqrt{2}\, t + \alpha) + d_v(t) \tag{6}$$

where $d_v(t)$ is a signal with (finite) significant energy only in the frequency band $[0, \ 3]$ $[rad/s]$.

Design a minimum complexity controller to achieve zero steady state error for the reference and the sinusoidal input disturbance at $\omega = \sqrt{2}$ whilst achieving a small error for the remainder of the input disturbance.

[1]Recall that a polynomial is dominated by its roots closest to the stability boundary .

Solutions to Solved Problem 7.4

Solved Problem 7.5. *Consider a plant having a nominal model given by*

$$G_o(s) = \frac{e^{-2s}}{s+1} = e^{-2s}\bar{G}_o(s) \tag{7}$$

Build a Smith predictor so that the settling time for a step reference is no more than 3 [s]. Assume that the reference and disturbances are step like signals

Solutions to Solved Problem 7.5

Solved Problem 7.6. *Contributed by - Alvaro Liendo, Universidad Tecnica Federico Santa Maria, Chile.*
Consider a nominal model given by:

$$G_o(s) = \frac{1}{(s+1)(s-2)} \tag{8}$$

Find a controller that stabilizes $G_o(s)$.

Solutions to Solved Problem 7.6

Solved Problem 7.7. *Contributed by - Alvaro Liendo, Universidad Tecnica Federico Santa Maria, Chile.*
Consider the same nominal model as in Problem 7.6. Find a controller that leads to:

$$A_{cl}(s) = (s+2)(s+3)(s+4)(s+5) \tag{9}$$

Solutions to Solved Problem 7.7

Solved Problem 7.8. *Contributed by - Alvaro Liendo, Universidad Tecnica Federico Santa Maria, Chile.*
Consider a nominal model given by

$$G_o(s) = \frac{3s+1}{(s+2)(s-3)} \tag{10}$$

The aim of the control law is to track a constant reference, and also to cancel the pole at $s = -2$ in $G_o(s)$. Find a suitable controller.

Solutions to Solved Problem 7.8

Solved Problem 7.9. *Contributed by - Alvaro Liendo, Universidad Tecnica Federico Santa Maria, Chile.*
Consider a nominal model given by

$$G_o(s) = \frac{1}{(s+1)^2} \tag{11}$$

Further assume that we need zero steady state error at zero frequency. We want to specify a third order polynomial $A_{cl}(s)$. From our analysis of the pole assignment methodology we know that not every third order $A_{cl}(s)$ can achieve stable closed loop. Find the family of stable third degree polynomials that yields a stable closed loop with zero steady state error at d.c.

Solutions to Solved Problem 7.9

Chapter 7 - Solutions to Solved Problems

Solution 7.1. *With experience, design problems such as this can be tackled by trial and error, using tools such as MATLAB* ***rltool****. However, a systematic approach gives a direct solution. Here we follow the latter approach.*

We begin by recalling that we can choose an arbitrary set of closed loop natural frequencies if the closed loop characteristic polynomial, $A_{cl}(s)$ *has degree at least equal to 2n-1 (the degree of* $A_o(s)$*, the plant nominal model denominator). However, since we want to force zero steady state error at d.c., an additional degree-of-freedom is needed. We are aiming for a minimum complexity controller; hence, we choose the degree of* $A_{cl}(s)$ *equal to 4.*

In addition, the roots of $A_{cl}(s)$ *should be to the left of* $s = -3$ *to ensure that the response time specification is met. Say we choose*

$$A_{cl}(s) = (s^2 + 6s + 9)(s + 4)(s + 5) = s^4 + 15s^3 + 83s^2 + 201s + 180 \tag{12}$$

We can now solve the Diophantine equation

$$\underbrace{(s-1)(s+2)}_{A_o(s)}\ \underbrace{s(s+\ell_0)}_{L(s)} + \underbrace{2}_{B_o(s)}\ \underbrace{(p_2s^2 + p_1s + p_0)}_{P(s)} = s^4 + 15s^3 + 83s^2 + 201s + 180 \tag{13}$$

The solution to this equation can found using the MATLAB function ***paq.m*** *distributed with the book and available on the web site.*

```
>>Ao=[1 1 -2];Am=[Ao 0];Bo=2;Acl=[1 15 83 201 180];
>> [Lm,P]=paq(Am,Bo,Acl); L=[Lm' 0];C=tf(P',L);
```

We finally obtain

$$P(s) = 35.5s^2 + 114.5s + 90 \tag{14}$$

$$L(s) = s^2 + 14s \tag{15}$$

$$C(s) = \frac{35.5s^2 + 114.5s + 90}{s^2 + 14s} \tag{16}$$

Solution 7.2. *We first note that the degree of* $A_{cl}(s)$ *should be at least 4. Furthermore, to enforce the specified dominance condition, we will define* $A_{cl}(s)$ *as*

$$A_{cl}(s) = (s + a)^2(s + 2a)^2; \quad \textit{where } a \in \{1, 8\} \tag{17}$$

7.2.1 *In this case we choose*[2] $A_{cl}(s) = (s + 1)^2(s + 2)^2$*. The corresponding diophantine equation becomes*

$$(s-4)(s+2))s(s+\ell_0) + (2s-2)(p_2s^2 + p_1s + p_0) = (s+1)^2(s+2)^2 \tag{18}$$

$$\Longrightarrow (s-4)s(s+\ell_0) + (2s-2)(\tilde{p}_1s + \tilde{p}_0) = (s+1)^2(s+2) = s^3 + 4s^2 + 5s + 2 \tag{19}$$

[2]Note that this choice of $A_{cl}(s)$ forces the cancellation of the plant pole at $s = -2$.

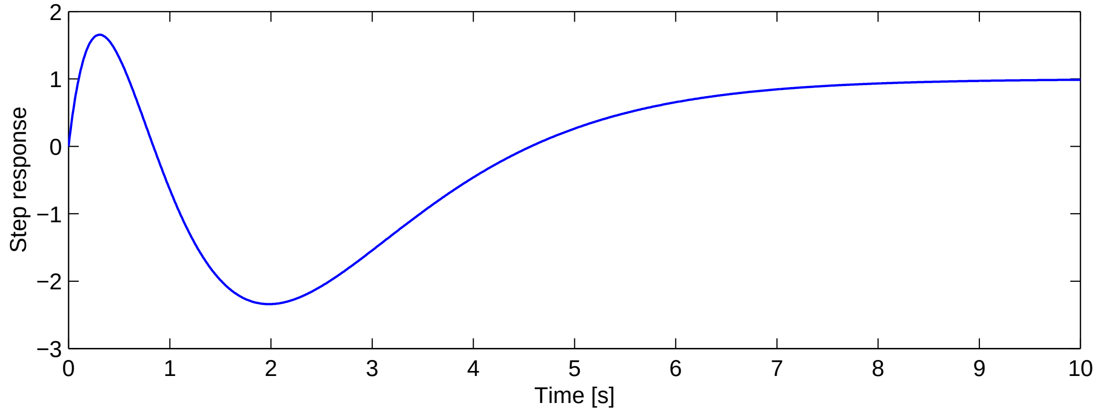

Figure 1: Response to a unit step reference with controller $C_1(s)$

Hence the controller transfer function is given by

$$C_1(s) = \frac{(s+2)(\tilde{p}_1 s + \tilde{p}_0)}{s(s+\ell_0)} = \frac{6.5s^2 + 12s - 2}{s^2 - 5s} \tag{20}$$

leading to the complementary sensitivity

$$T_{o1}(s) = \frac{13s^2 - 15s + 2}{s^3 + 4s^2 + 5s + 2} \tag{21}$$

The response to a unit step reference for this design is shown in Figure 1.

7.2.2 In this case[3] $A_{cl}(s) = (s+8)^2(s+16)^2$ *and the diophantine equation is*

$$(s-4)(s+2))s(s+\ell_0) + (2s-2)(p_2 s^2 + p_1 s + p_0) = (s+8)^2(s+16)^2 \tag{22}$$

Using the MATLAB function ***paq*** *we obtain*

$$C_2(s) = \frac{1326s^2 + 856s - 8192}{s^2 - 2602s} \tag{23}$$

leading to the complementary sensitivity

$$T_{o2}(s) = \frac{2652s^3 - 4364s^2 - 1.4670s + 1.6380}{s^4 + 48s^3 + 832s^2 + 6144s + 1.6380} \tag{24}$$

The corresponding response to a unit step reference is shown in Figure 2

[3] Note that this choice of $A_{cl}(s)$ avoids cancellation of any of the plant poles.

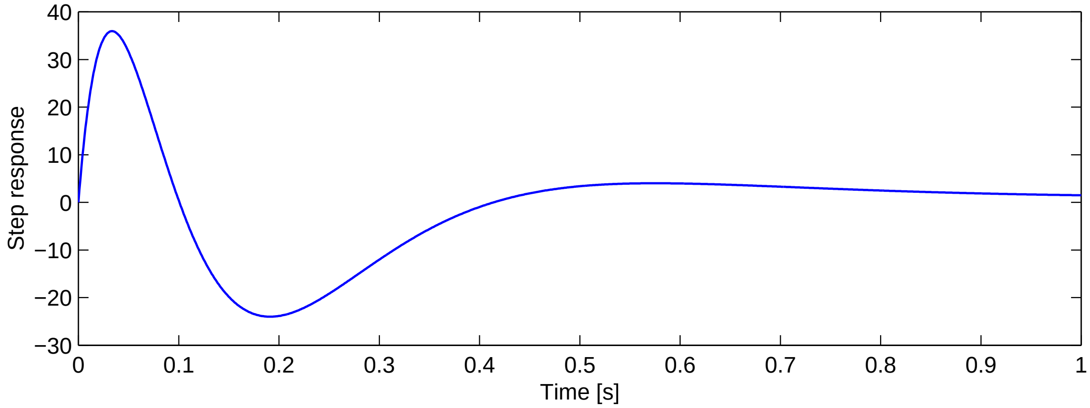

Figure 2: Response to a unit step reference with controller $C_2(s)$

7.2.3 *When comparing the performance of both designs it is evident that the settling time is much smaller in the second case, as expected, since the dominant closed loop poles are much faster than in the first design. However it is also clear that this faster response comes at a price: namely very large overshoot and very large undershoot. In Chapter 8 of the book it is shown that this sort of performance originates from fundamental limitations associated with the structure and location of poles and zeros of this particular plant.*

Solution 7.3. *We note that, for a second order plant, a controller with integration requires (for arbitrary selection of the closed loop poles) the specification of a fourth order $A_{cl}(s)$. This, applied to our plant, would have yielded a second order controller. However, the PI controller is only of first order. Thus a controller of this structure does not allow arbitrary specification of the closed-loop poles. However, it is possible to locate the poles provided they are constrained.*

We observe that with $G_o(s)$ and $C(s)$ above, the complementary sensitivity is given by

$$T_o(s) = \frac{s + 0.7}{s^3 + 2s^2 + 2s + 0.7} \tag{25}$$

Assume that we specify

$$A_{cl}(s) = (s^3 + 2s^2 + 2s + 0.7)X(s) \tag{26}$$

where $X(s)$ is an arbitrary first order stable polynomial, then the corresponding Diophantine equation

$$(s+1)^2 \underbrace{s(s+\ell)}_{L(s)} + \underbrace{p_2 s^2 + p_1 s + p_0}_{P(s)} = (s^3 + 2s^2 + 2s + 0.7)X(s) \tag{27}$$

would have as solutions

$$P(s) = (s + 0.7)X(s) \tag{28}$$
$$L(s) = sX(s) \tag{29}$$

This suggests that there are an infinite number of pole assignment synthesis which could yield a controller of reduced complexity – provided that part of the closed-loop characteristic polynomial is chosen to satisfy the special constraints associated with using a reduced order controller.

Solution 7.4. *Since the plant is of order 2, the minimum degree of $A_{cl}(s)$ is 3. However we require to force integration in the controller to achieve zero steady state error for constant references. This will take the degree of $A_{cl}(s)$ to 4. Furthermore, we need to have zero steady state error for a sinusoidal disturbance of frequency $\sqrt{2}$ [rad/s]. This requirement is equivalent to requiring perfect inversion at $\omega = \sqrt{2}$ [rad/s], and, therefore, to have controller poles at $s = \pm j\sqrt{2}$. This takes the degree of $A_{cl}(s)$ to 6.*

To choose the sixth degree polynomial $A_{cl}(s)$ we consider the bandwidth of the disturbance component $d_v(t)$. This suggests that the dominant pole should be to the left of -3. Say we choose $A_{cl}(s)$ dominated by the factor[4] $(s^2 + 8s + 20)(s + 4)$. Thus, a possible choice is

$$A_{cl}(s) = (s^2 + 8s + 20)(s + 4)(s + 5)(s + 6)(s + 7) \tag{30}$$

The corresponding Diophantine equation becomes

$$(s+2)(s+4)\underbrace{s(s^2+2)(s+\ell)}_{L(s)} + 4\underbrace{(s+4)(\tilde{p}_3 s^3 + \tilde{p}_2 s^2 + \tilde{p}_1 s + p_0)}_{P(s)} = (s^2+8s+20)(s+4)(s+5)(s+6)(s+7) \tag{31}$$

After cancelling the factor $(s + 4)$, we use the MATLAB function ***paq*** *via the following code*

```
>> Ax=conv([1 2],[1 0 2 0]);Bx=4;Aclx=conv([1 8 20], poly([-5 -6 -7]));
>>[Lx,Px]=paq(Ax,Bx,Aclx)
>>P=conv(Px',[1 4]); L=conv(Lx',[1 0 2 0]);C=tf(P,L);
```

leading to the controller

$$C(s) = \frac{55.25s^4 + 564.5s^3 + 2305s^2 + 4774s + 4200}{s^4 + 24s^3 + 2s^2 + 48s} = \frac{55.25s^4 + 564.5s^3 + 2305s^2 + 4774s + 4200}{s(s^2+2)(s+24)} \tag{32}$$

We observe that the controller has poles at $s = 0$ and $s = \pm j\sqrt{2}$, as expected. To evaluate the performance of the design regarding disturbance rejection we analyze the magnitude of the frequency response of the input sensitivity S_{io}. This can be achieved with the following MATLAB code[5]

```
>> Go=tf(4,[1 6 8]);Sio=minreal(Go/(1+Go*C));
>>w=logspace(-1,1,1000); h=freqresp(Sio,w);
>>subplot(211); semilogx(w, 20*log10(abs(h(1,:))));
```

This yields the plot shown in Figure 3

From Figure 3 we observe that the input sensitivity attains its maximum value at approximately $\omega = 5.5$ [rad/s] and this maximum is equal to -20 [dB]. We also see that a disturbance in the band $[0,\ 3]$ [rad/s] is attenuated by more than 90%.

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[4]Note that by including a closed-loop pole at $s = -4$ we are forcing the cancellation of the plant pole at that location.

[5]We use the previously calculated transfer function $C(s)$.

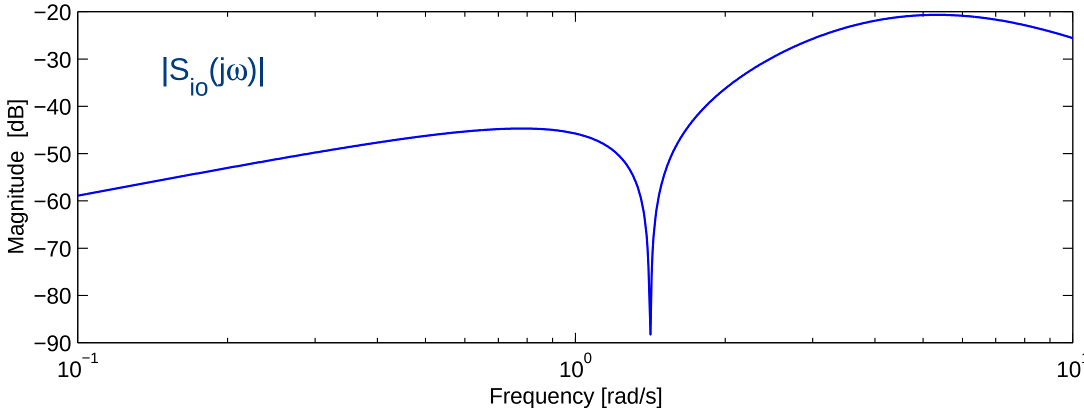

Figure 3: Input sensitivity frequency response

Solution 7.5. *We refer to section §7.4 of the book. We see that we can apply the pole assignment technique to the rational part of* $G_o(s)$.

Since the plant has a 2 $[s]$ *pure delay, we cannot aim for a settling time less than that if we want a robust closed loop. Say we choose the settling time equal to 3* $[s]$*. We recall that the settling time is usually defined as the pure delay plus four times the dominant time constant. This implies that the dominant pole should be located to the left of* -4*; this will yield a dominant time constant equal to* 0.25 $[s]$*. Say we then choose (for the rational part i.e.,* $\bar{G}_o(s)$*)*

$$A_{cl}(s) = s^2 + 8s + 20 \tag{33}$$

Then the corresponding Diophantine equation is

$$(s+1)\underbrace{s}_{L(s)} + 1 \cdot \underbrace{(p_1 s + p_0)}_{P(s)} = s^2 + 8s + 20 \tag{34}$$

Then $C(s)$*, in Figure 7.1 of the book, is given by*

$$C(s) = \frac{7s+20}{s} \tag{35}$$

And the resultant complementary sensitivity is given by:

$$T_o(s) = \frac{7s+20}{s^2+8s+20} e^{-2s} \tag{36}$$

Solution 7.6. $A_{cl}(s)$ *can be arbitrarily specified if its degree is, at least, three. Say we choose*

$$A_{cl}(s) = (s+2)(s+3)(s+4) \tag{37}$$

This leads to a controller of the form

$$C(s) = \frac{p_1 s + p_0}{s + \ell_0} \tag{38}$$

The pole assignment equation is then

$$A_o(s)L(s) + B_o(s)P(s) = (s+2)(s+3)(s+4) \tag{39}$$
$$(s+1)(s-2)(s+\ell_0) + (p_1 s + p_0) = (s+2)(s+3)(s+4) \tag{40}$$
$$s^3 + (\ell_0 - 1)s^2 + (p_1 - \ell_0 - 1)s + (p_0 - 2\ell_0) = s^3 + 9s^2 + 26s + 24 \tag{41}$$

This polynomial identity leads to equations

$$\ell_0 - 1 = 9 \tag{42}$$
$$p_1 - \ell_0 - 1 = 26 \tag{43}$$
$$p_0 - 2\ell_0 = 24 \tag{44}$$

Solving yields the controller

$$C(s) = \frac{38s + 44}{s + 10} \tag{45}$$

To solve this kind of problem we can also use the MATLAB function ***paq.m****, as shown in the following MATLAB code:*

```
>>[L,P]=paq([1 -1 -2],1,[1 9 26 24])
```

In this command the first argument is $A_o(s)$*, the second is* $B_o(s)$ *and the third is* $A_{cl}(s)$*. The function returns the polynomials* $P(s)$ *and* $L(s)$

Solution 7.7. *The degree of* $A_{cl}(s)$ *is four and the degree of* $A_o(s)$ *is two. Hence* $L(s)$ *should have a degree equal to two. Then*

$$C(s) = \frac{p_2 s^2 + p_1 s + p_0}{s^2 + \ell_1 s + \ell_0} \tag{46}$$

The corresponding pole assignment equation becomes

$$A_o(s)L(s) + B_o(s)P(s) = (s+2)(s+3)(s+4)(s+5) \tag{47}$$
$$(s+1)(s-2)(s^2 + \ell_1 s + \ell_0) + (p_2 s^2 + p_1 s + p_0) = (s+2)(s+3)(s+4)(s+5) \tag{48}$$
$$s^4 + (\ell_1 - 1)s^3 + (p_2 - \ell_1 + \ell_0 - 2)s^2 + (p_1 - \ell_0 - 2\ell_1)s + (p_0 - 2\ell_0) = s^4 + 14s^3 + 71s^2 + 154s + 120 \tag{49}$$

This polynomial identity leads to the equations

$$\ell_1 - 1 = 14 \tag{50}$$
$$p_2 - \ell_1 + \ell_0 - 2 = 71 \tag{51}$$
$$p_1 - \ell_0 - 2\ell_1 = 154 \tag{52}$$
$$p_0 - 2\ell_0 = 120 \tag{53}$$

The solution of these equations is

$$\ell_1 = 15 \tag{54}$$
$$p_2 = 73 + \ell_1 - \ell_0 = 88 - \ell_0 \tag{55}$$
$$p_1 = 154 + 2\ell_1 + \ell_0 = 184 + \ell_0 \tag{56}$$
$$p_0 = 120 + 2\ell_0 \tag{57}$$
$$\tag{58}$$

We thus observe that there is an infinite number of solutions. Every choice of ℓ_0 leads to a different stabilizing controller, for example, to force integration in the controller we can choose $\ell_0 = 0$

$$C(s) = \frac{88s^2 + 184s + 120}{s^2 + 15s} \tag{59}$$

In MATLAB we can solve this problem with the command line

```
>>[L,P]=paq([1 -1 -2],1,[1 14 71 154 120])
```

This leads to

$$C(s) = \frac{272s + 296}{s^2 + 15s + 88} \tag{60}$$

*This is equivalent to choosing $\ell_0 = 88$. The algorithm in **paq.m** has been designed to yield a minimum degree $P(s)$.*

Solution 7.8. *The minimum degree of $A_{cl}(s)$ is four since we need to force integration in the controller, then*

$$C(s) = \frac{p_2 s^2 + p_1 s + p_0}{s(s + \ \ell_1)} \tag{61}$$

Since $(s + 2)$ is cancelled if and only if $(s + 2)$ is a factor of $A_{cl}(s)$, we choose

$$A_{cl}(s) = (s + 2)(s + 3)(s + 4)(s + 5) \tag{62}$$

The pole assignment equation then becomes

$$A_o(s)L(s) + B_o(s)P(s) = (s+2)(s+3)(s+4)(s+5) \tag{63}$$
$$(s+2)(s-3)(s+\ell_1)s + (3s+1)(p_2 s^2 + p_1 s + p_0) = (s+2)(s+3)(s+4)(s+5) \tag{64}$$

In this polynomial identity we note that $(s+2)$ has to be a factor of $P(s)$ (a plant pole can only be cancelled by a controller zero). Thus we define

$$(s+2)(\tilde{p}_1 s + \tilde{p}_0) = p_2 s^2 + p_1 s + p_0 \tag{65}$$

The pole assignment equation then simplifies as follows:

$$(s-3)(s+\ell_1)s + (3s+1)(\tilde{p}_1 s + \tilde{p}_0) = (s+3)(s+4)(s+5) \tag{66}$$
$$s^3 + (3\tilde{p}_1 + \ell_1 - 3)s^2 + (\tilde{p}_1 + 3\tilde{p}_0 - 3\ell_1)s + \tilde{p}_0 = s^3 + 12s^2 + 47s + 60 \tag{67}$$

This leads to equations

$$3\tilde{p}_1 + \ell_1 - 3 = 12 \tag{68}$$
$$\tilde{p}_1 + 3\tilde{p}_0 - 3\ell_1 = 47 \tag{69}$$
$$\tilde{p}_0 = 60 \tag{70}$$

The solution is $\tilde{p}_0 = 60$, $\tilde{p}_1 = -\frac{44}{5}$ and $\ell_1 = \frac{207}{5}$. Finally the controller is

$$C(s) = \frac{(s+2)(-44s+300)}{s(5s+207)} \tag{71}$$

Solution 7.9. *Define*

$$A_{cl}(s) = s^3 + a_2 s^2 + a_1 s + a_0 \tag{72}$$

Then, the degree of $L(s)$ must be one and, to satisfy the steady state requirement, it must have integration. Hence the controller has the form

$$C(s) = \frac{p_1 s + p_0}{s} \tag{73}$$

The pole assignment equation is

$$A_o(s)L(s) + B_o(s)P(s) = s^3 + a_2 s^2 + a_1 s + a_0 \tag{74}$$
$$s(s+1)^2 + (p_1 s + p_0) = s^3 + a_2 s^2 + a_1 s + a_0 \tag{75}$$
$$s^3 + 2s^2 + (p_1+1)s + p_0 = s^3 + a_2 s^2 + a_1 s + a_0 \tag{76}$$

From this we see that $a_2 = 2$ *assures integration in the controller. Furthermore, since* $A_{cl}(s)$ *has to be stable, conditions to satisfy this requirement can be obtained using Routh's Algorithm applied to the polynomial* $A_{cl}(s) = s^3 + 2s^2 + a_1 s + a_0$

The algorithm yields A_{cl} *stable if and only if*

$$2a_1 > a_0 \tag{77}$$

$$a_0 > 0 \tag{78}$$

Finally, the family of polynomials satisfying both, stability and integration, is

$$\mathcal{A} = \{s^3 + 2s^2 + a_1 s + a_0 \in \mathbb{R}_3[s] : a_0 > 0, a_1 > 0, 2a_1 - a_0 > 0\} \tag{79}$$

where $\mathbb{R}_3[s]$ *is the ring of polynomials in* s *of degree less or equal to three, with real coefficients.*
