Chapter 8 - Solved Problems

Solved Problem 8.1. *Consider the system described in Solved Problem 5.4 of Chapter 5. Do you think control of this system would be easy or difficult.*

Solutions to Solved Problem 8.1

Solved Problem 8.2. *A plant has a nominal model given by*

$$G_o(s) = \frac{-s+2}{(s+1)^2} \tag{1}$$

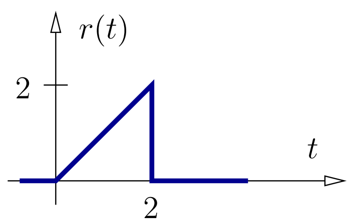
Assume that a feedback controller is designed such that all the closed loop poles have real parts less than -1, so as to yield zero steady state errors for constant references and disturbances (i.e., the controller has a pole at the origin). Discuss the impact of the plant NMP zero on the time domain fundamental limitation relating to controller output $u(t)$ for a unit step reference?.

Solutions to Solved Problem 8.2

Solved Problem 8.3. *In a feedback control loop with integration and a plant with transfer function*

$$G_o(s) = \frac{2}{(s+2)^2} \tag{2}$$

the reference is as shown in the figure. Will the output overshoot the reference at any time?



Solutions to Solved Problem 8.3

Solved Problem 8.4. *The nominal model of an unstable system is given by*

$$G_o(s) = \frac{1}{s-1} \tag{3}$$

This plant has to controlled with zero steady state error for constant references.

8.4.1 *Synthesize a PI controller such as the closed loop poles are located at -0.5 and -1*

8.4.2 *Compute the closed loop response to a unit step reference. Comment.*

Solutions to Solved Problem 8.4

Solved Problem 8.5. *Figure 8.5 in the book depicts the control error for a unit step reference for a system having zeros on the imaginary axis. Note that the error, $e(t)$ is significantly larger than unity. This is a necessary and sufficient condition for the plant output to exhibit undershoot. In this case, a large undershoot appears. Note however that if the complementary sensitivity had been*

$$T_o(s) = \frac{5s^2+1}{s^3+3s^2+3s+1} \tag{4}$$

then, no undershoot arises.

Show that for a given system with zeros on the imaginary axis, the step response will always exhibit undershoot provided that the zeros have a magnitude sufficiently small.

Solutions to Solved Problem 8.5

Chapter 8 - Solutions to Solved Problems

Solution 8.1. *We saw in the solution to Solved Problem 5.4 that the controller had to be unstable. However, the situation is compounded by the fact that there is necessarily two real unstable poles having magnitude greater than a nonminimum phase zero. The guidelines presented in chapter 8 of the book suggest that the bandwidth should be greater than any real unstable pole and less than any real nonmimimum phase zero. This is clearly impossible here. Indeed, such a system would be virtually impossible to control in a practical sense.*

Solution 8.2. *Given the closed loop pole specification, we observe that the closed loop is internally stable, hence $T_o(2) = 1$. We next have that*

$$U(s) = S_{uo}(s)R(s) = \frac{T_o(s)}{G_o(s)}R(s) \tag{5}$$

Also, since the controller has a pole at the origin then, for a unit step reference, $u(\infty) = G_o(0)^{-1}$. Define

$$u_\Delta(t) = u(t) - u(\infty) \tag{6}$$

Then $U_\Delta(s)$ converges for all $\Re\{s\} > -1$. Therefore

$$U_\Delta(s) = \left[\frac{T_o(s)}{G_o(s)} - \frac{1}{G_o(0)}\right]\frac{1}{s} = \int_0^\infty u_\Delta(t)e^{-st}\,dt \tag{7}$$

And evaluating at $s = 2$, we obtain

$$U_\Delta(0) = 0 = \int_0^\infty u_\Delta(t)e^{-2t}\,dt \tag{8}$$

The above equation says that the controller output must overshoot and undershoot its stationary value.

Solution 8.3. *The reference has Laplace transform given by*

$$R(s) = \frac{1 - e^{-2s} - 2s}{s^2} \tag{9}$$

Note that $\lim_{s\to 0} sR(s) = 0$.
Furthermore, the error is given by

$$E(s) = S_o(s)R(s) = s\tilde{S}_o(s)R(s) \qquad \forall\Re\{s\} \geq 0 \tag{10}$$

where $\tilde{S}_o(s)$ is a stable rational, strictly proper transfer function.
Then

$$E(0) = \int_0^\infty e(t)\,dt = \lim_{s\to 0} \tilde{S}_o(s)sR(s) = 0 \tag{11}$$

Hence, since the integral of the error is zero, it is evident that the error must be negative during at least one nonzero time interval, i.e., ***the output overshoots the reference during those time intervals****. This result holds for all* $G_o(s)$*.*

Solution 8.4.

8.4.1 *This can be solved as a pole assignment problem using* $A_{cl}(s) = (s+0.5)(s+1) = s^2 + 1.5s + 0.5$*. This leads to*

$$\underbrace{s}_{L(s)}(s-1) + \underbrace{as+b}_{P(s)} = \underbrace{s^2 + 1.5s + 0.5}_{A_{cl}(s)} \tag{12}$$

Hence, the controller transfer function is given by

$$C(s) = \frac{2.5s + 0.5}{s} \tag{13}$$

8.4.2 *The complementary sensitivity function,* $T_o(s)$*, can then be computed using the following MATLAB code*

```
≫Go=tf(1,[1 -1]);C=tf([2.5 0.5],[1 0]);
≫To=minreal(Go*C/(1+Go*C));
```

The step response can then be obtained with the MATLAB command ***step(To)****. The result is shown in Figure 1. A large overshoot (approximately* 56 *%) can be observed. This was expected before we built the controller since equation (8.6.27) from the book applies, with* $\alpha = 0.5$ *and* $\eta_o = 1$*. Therefore, the error must become negative (*$y > r$*) during nonzero time interval(s).*

However there is an additional problem. After we synthesize the controller, we realize that the open loop has a zero located at $z_o = -0.2$*, this is also located to the right of* $-\alpha$*. This implies that equation (8.6.26) from the book applies, i.e.,*

$$\int_0^\infty e(t)e^{0.2t}\,dt = \frac{1}{z_o} = -5 \tag{14}$$

As a consequence, this zero also forces overshoot in the step response.

Solution 8.5. *Assume that the system has zeros at* $s = \pm j\omega_o$*, then the complementary sensitivity can always be expressed as*[1]

[1] This is due to the fact that these zeros must not be cancelled. See subsection 8.6.3 of the book.

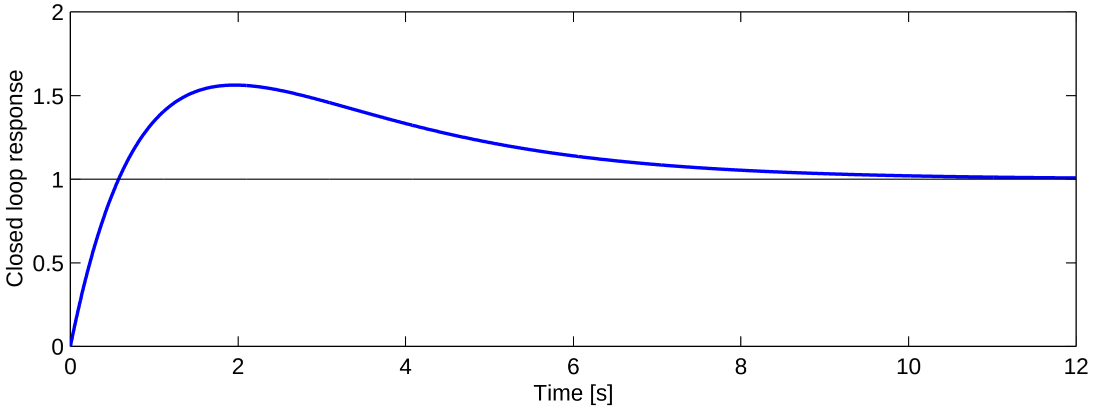


Figure 1: Closed-loop response to a unit step reference

$$T_o(s) = \left(\frac{s^2}{\omega_o^2} + 1\right)\tilde{T}_o(s) \tag{15}$$

Therefore the closed loop response for a unit step reference is given by

$$Y(s) = \frac{1}{\omega_o^2}\underbrace{s\tilde{T}_o(s)}_{Y_1(s)} + \underbrace{\tilde{T}_o(s)\frac{1}{s}}_{Y_2(s)} = \frac{1}{\omega_o^2}Y_1(s) + Y_2(s) \tag{16}$$

Note that $Y_1(s)$ *has all its poles in the* ***open*** *LHP, thus its Laplace transform converges for all* $\Re\{s\} \geq 0$. *We then have*

$$Y_1(0) = 0 = \int_0^\infty y_1(t)\,dt \tag{17}$$

The above equation says that $y_1(t)$ *must be negative during some nonzero time interval(s). Say that* $y_1(t) < 0$ *for* $t \in (t_1, t_2)$. *This implies that there exists a positive constant* m_1 *satisfying*

$$\min_{t\in(t_1,t_2)} y_1(t) = -m_1 < 0 \tag{18}$$

Furthermore, assume that there exists a positive constant M_2 *such that*

$$\max_{t\in(t_1,t_2)} y_2(t) = M_2 \tag{19}$$

Then, the total response $y(t)$ *will be negative provided*

$$\frac{m_1}{-\omega_o^2} + M_2 < 0 \tag{20}$$

This shows that for ω_o *small enough,* $y(t)$ *can be made negative sometime in the interval* (t_1, t_2).